Computing the Difficulty of Critical Bootstrap Percolation Models is NP-hard

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Abstract

Bootstrap percolation is a class of cellular automata with random initial state. Two-dimensional bootstrap percolation models have three universality classes, the most studied being the 'critical' one. For this class the scaling of the quantity of greatest interest — the critical probability — was determined by Bollobás, Duminil-Copin, Morris and Smith [4] in terms of a combinatorial quantity called 'difficulty', so the subject seemed closed up to finding sharper results. In this paper we prove that computing the difficulty of a critical model is NP-hard and exhibit an algorithm to determine it, in contrast with the upcoming result of Balister, Bollobás, Morris and Smith [3] on undecidability in higher dimensions. The proof of NP-hardness is achieved by a reduction to the Set Cover problem.

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1 Introduction

1.1 Background

Bootstrap percolation is a class of cellular automata whose first representative was introduced in 1979 by Chalupa, Leath and Reich [7] in statistical physics. Further applications to several other areas have been considered, namely dynamics of the Ising model, kinetically constrained models for the glass transition, abelian sandpiles and others (see a recent review of Morris [18] for more information). Given a set $A \subset \mathbb{Z}^d$ or $(\mathbb{Z}/n\mathbb{Z})^d$ of initially infected sites, more vertices become infected at each discrete time step following a deterministic monotone local rule invariant in time and space, while infections never heal. More precisely, let us introduce the broadest framework introduced by Bollobás, Smith and Uzzell [6].[1]

A bootstrap percolation model is given by a finite set $\mathcal{U}$, called *update family*, of finite subsets of $\mathbb{Z}^d \setminus \{0\}$, called *rules*. For an initial set of infected sites $A = A_0 \subset \mathbb{Z}^d$ we define

$$A_{t+1} = A_t \cup \{x \in \mathbb{Z}^d \ : \ \exists U \in \mathcal{U},\ x + U \subset A_t\}$$

and $[A]$ is the closure of A with respect to this operation. We will only discuss the most studied case, where A is taken at random according to the product Bernoulli measure $\mathbb{P}_p$, so that each site is initially infected with probability p. Equipped with this measure, the model exhibits a phase transition at

$$p_c = \inf\{p \in [0,1] \ : \ \mathbb{P}_p(0 \in [A]) = 1\}.$$

The model is defined identically on tori $(\mathbb{Z}/n\mathbb{Z})^d$ by setting

$$p_c(n) = \inf\{p \in [0,1] \ : \ \mathbb{P}_p([A] = (\mathbb{Z}/n\mathbb{Z})^d) \geqslant 1/2\}.$$

Although for some concrete models higher dimensions have been understood and some general universality conjectures have been put forward in [2, Conjecture 16] and [4, Conjecture 9.2], we will restrict our attention to the 2-dimensional case. The results of Bollobás, Smith and Uzzell [6] and Balister, Bollobás, Przykucki and Smith [2] combined establish that all bootstrap percolation models can be partitioned (by a simple procedure) into 3 "universality classes" with the similar scaling of $p_c(n)$. In order to define these we need some notation. For a direction $u \in S^1 = \{x \in \mathbb{R}^2 \ : \ \|x\|_2 = 1\}$ we denote by

$$\mathbb{H}_u = \{x \in \mathbb{Z}^2 \ : \ \langle x, u\rangle < 0\}$$

[1] Earlier partly non-rigorous considerations of a more restricted class of models can be found in the works of Gravner and Griffeath [10, 11] from the 90's.

the open half-plane directed by u and by

$$l_u = \{x \in \mathbb{Z}^2,\ \langle x, u\rangle = 0\}$$

the line passing through 0 perpendicular to u. A direction u is *unstable* if there exists $U \in \mathcal{U}$ such that $U \subset \mathbb{H}_u$ and *stable* otherwise. It is easy to see [6] that the unstable directions form a finite union of open intervals in S^1 and the partition into universality classes is in terms of these directions.

- $\mathcal{U}$ is supercritical if there exists an open semi-circle of unstable directions, in which case $p_c(n) = n^{-\Theta(1)}$.
- $\mathcal{U}$ is critical if it is not supercritical and there exists a semi-circle with a finite number of stable directions, in which case $p_c(n) = (\log n)^{-\Theta(1)}$.
- $\mathcal{U}$ is subcritical otherwise (if each semi-circle contains infinitely many stable directions), in which case $p_c > 0$.

The behavior of supercritical models is dominated by the study of finite infected sets with infinite closure, while subcritical ones are more closely related to percolation. The most studied models are critical ones, to which the archetypal example of bootstrap percolation belongs — the 2-neighbor model, in which a site becomes infected if it has at least two infected neighbors. This is the first model for which the universality result above (and more) was established — by Aizenman and Lebowitz [1]. They realized that the dynamics is dominated by a bottleneck — creating an infected "droplet" of a certain "critical" size, which can then easily grow out to infinity, and proved that for this model $p_c(n) = \Theta(1/\log n)$. In a substantial breakthrough Holroyd [15] determined the asymptotic location of the sharp threshold and since then much sharper results have been proved [12, 14]:

$$p_c = \frac{\pi^2}{18 \log n} - \frac{\Theta(1)}{(\log n)^{3/2}}.$$

Such sharp or sharper bounds have been obtained for a handful of other specific models [5,8,9], but still remain open in general. However, the level of precision of the Aizenman-Lebowitz result was established in full generality for critical models by Bollobás, Duminil-Copin, Morris and Smith [4]. They introduce the following key notion of "difficulty".

Definition 1.1 (Definition 1.2 of [4][2])**.** Let $\mathcal{U}$ be a critical model and u be an isolated stable direction. We then define the difficulty of u, $\alpha(u)$, to be the

[2] The definition we give is formally different from the one in [4], but the two are easily seen to be equivalent.

minimum cardinality of a set $Z \subset \mathbb{Z}^2 \setminus \mathbb{H}_u$ such that $[\mathbb{H}_u \cup Z] \setminus \mathbb{H}_u$ is infinite. For unstable directions u we set $\alpha(u) = 0$ and for non-isolated stable ones we set $\alpha(u) = \infty$. The difficulty of $\mathcal{U}$ is

$$\alpha = \inf_{C \in \mathcal{C}} \sup_{u \in C} \alpha(u), \tag{1}$$

where $\mathcal{C}$ is the set of open semi-circles of S^1.

The result of [4] then states[3]

$$p_c(n) = \frac{(\log \log n)^{\mathcal{O}(1)}}{(\log n)^{1/\alpha}}.$$

1.2 Results

Motivated by the notion of critical densities, corresponding to difficulties, but for subcritical models, introduced by the first author in [13], which are rather complicated in nature, we examine how hard it is to actually determine α given the update family. Somewhat surprisingly, given the simple Definition 1.1, it turns out that difficulties are difficult to determine. Another important motivation comes from a related phenomenon in higher dimensions noticed by Balister, Bollobás, Morris and Smith [3] and announced prior to our work. There, an even more striking result emerges owing to the richness of supercritical 2-dimensional models: the exponent determining the scaling of $p_c(n)$ is uncomputable. In two dimensions we prove the following less deterring result.

Theorem 1.2. *The problem of computing the difficulty α of a critical bootstrap percolation family $\mathcal{U}$ is NP-hard.*

Remark 1.3. The same result holds for the bilateral difficulty β introduced by Martinelli, Morris and Toninelli [17] relevant for the kinetically constrained model associated to a critical bootstrap percolation model. Indeed, it suffices to consider the symmetrised family $\mathcal{U}' = \{U, -U \ : \ U \in \mathcal{U}\}$ with $\mathcal{U}$ from the proof of the theorem and proceed as in Section 2. Then one obtains $\beta(\mathcal{U}') = \alpha(\mathcal{U}') = \alpha(\mathcal{U})$.

This result is proved by a fairly technical reduction to the SET COVER decision problem in Section 2. However, in order for this result to be meaningful, we need to make sure that there is *some* algorithm to compute the

[3] They actually give matching bounds up to a constant factor, which requires dividing critical models into two subclasses with different logarithmic factors.

difficulties. This is all the more necessary in view of the result of [3], stating that in higher dimensions this is not the case. Thus, it is important to have some bound on the complexity of the problem, as provided by the next theorem. The algorithm in question is fully explicit and given in Section 3.

Theorem 1.4. [4] *There exists an algorithm which, given a critical bootstrap percolation family $\mathcal{U}$, computes its difficulty α.*

Remark 1.5. In fact, it is not hard to check that our algorithm runs in time

$$|\mathcal{U}|^2 \cdot 2^{D^2(1+o(1)))} = \exp(\mathcal{O}(D^2)),$$

where D is defined in (4). This bound is clearly as sharp as a bound in terms of D only can be, since the input can be that large.

2 NP-hardness: proof of Theorem 1.2

In this section we prove Theorem 1.2 by providing a reduction from Set Cover to 2D Critical-Bootstrap Difficulty. For the Set Cover problem we consider a *universe* $\{1,\dots,N\}$ and a collection $\mathcal{S}$ of subsets of the universe and assume that $|\mathcal{S}| \geqslant 4$ and $N \geqslant 4$. The Set Cover problem asks for determining the minimum cardinality of a subset of $\mathcal{S}$ which covers the universe. It is one of the first NP-complete problems described by Karp [16].

We fix an instance

$$\mathcal{S} = \{S_i \;:\; i \in \mathbb{Z}, 1 \leqslant i \leqslant |\mathcal{S}|\}.$$

Our goal is to define a critical bootstrap percolation family whose difficulty α is (up to a simple transformation) the solution to Set Cover. Let the set of rules associated to $\mathcal{S}$ be

$$\mathcal{U} = \{U_0, U_1\} \cup \{U^k_{i,j} \;:\; 1 \leqslant i \leqslant |\mathcal{S}|,\, 1 \leqslant k \leqslant |\mathcal{S}|^2,\, i,k \in \mathbb{Z},\, j \in S_i\},$$

where

$$U_0 = \left\{(-k,0),(0,-k) \;:\; 1 \leqslant k \leqslant N|\mathcal{S}|^2\right\},$$
$$U_1 = \left\{(+k,0),(0,-k) \;:\; 1 \leqslant k \leqslant N|\mathcal{S}|^2\right\}$$

and the rules $U^k_{i,j}$ defined as follows share a large portion of their structure (see Figure 1).

$$\begin{aligned} T &= \left\{(0,-y) \;:\; 1 \leqslant y \leqslant N \cdot |\mathcal{S}|^2\right\}, \\ W &= \{(x,0) \;:\; 1 \leqslant x \leqslant |\mathcal{S}|^2\} \cup \{(l \cdot |\mathcal{S}|, 1) \;:\; 1 \leqslant l \leqslant |\mathcal{S}|\}, \\ U^k_{i,j} &= T \cup \big((W \cup \{(i \cdot |\mathcal{S}|, 2)\}) - (k + (N+j) \cdot |\mathcal{S}|^2, 0)\big). \end{aligned}$$

[4] This result was proved independently by Balister, Bollobás, Morris and Smith [3].

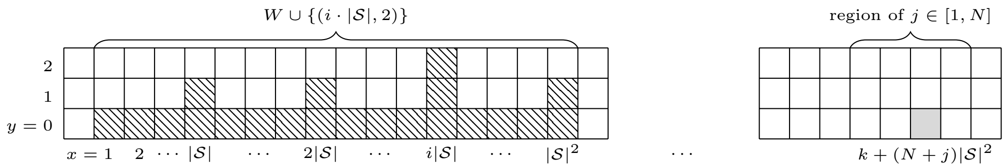

Figure 1: A visualisation of $(U^k_{i,j} \setminus T) + (k + (N + j)|\mathcal{S}|^2, 0)$; the shaded cell indicates where the origin is shifted to.

It is easy to check that the only isolated stable direction is $u = (0, 1)$, while $(S^1 \cap \mathbb{H}_{(0,1)}) \cup \{(1, 0), (-1, 0)\}$ are the only other stable ones, so that $\mathcal{U}$ is critical and $\alpha(\mathcal{U}) = \alpha_u$. We thus focus on this direction. Let $M \subset \{1, \ldots, |\mathcal{S}|\}$ be an optimal solution to the Set Cover problem given by $\mathcal{S}$ i.e. a set of minimal size such that

$$\bigcup_{i \in M} S_i = \{1, \ldots, N\}.$$

Setting

$$Z_0 = W \cup \{(i \cdot |\mathcal{S}|, 2) \ : \ i \in M\}$$

proves that

$$\alpha_u \leqslant |Z_0| = |W| + |M| = |\mathcal{S}|^2 + |\mathcal{S}| + |M|. \tag{2}$$

Indeed, using the rules $U^k_{i,j}$ for $i \in M$ and all j, k, one infects all sites in

$$\left[1 + (N + 1) \cdot |\mathcal{S}|^2, (2N + 1) \cdot |\mathcal{S}|^2\right] \times \{0\},$$

since M is a cover, and those are enough to infect l_u using U_0 and U_1.

For any $Z \subset \mathbb{Z}^2$ we denote

$$\bar{Z} := [Z \cup \mathbb{H}_u] \setminus \mathbb{H}_u. \tag{3}$$

To prove that (2) is actually an equality, we suppose that there exists a set $Z \subset \mathbb{Z}^2 \setminus \mathbb{H}_u$ for which $|\bar{Z}| = \infty$ and $|Z| < |Z_0|$. Fix a minimal such set Z. If there exists $p \in \mathbb{Z}^2 \setminus \mathbb{H}_u$ such that one of $p + U_0$ and $p + U_1$ is a subset of $Z \cup \mathbb{H}_u$, then we have a contradiction with the assumption that $|Z| < |Z_0|$. However, some of the rules must be applicable to $Z \cup \mathbb{H}_u$ and therefore there exists $p \in \mathbb{Z}^2 \setminus \mathbb{H}_u$ such that $p + W \subset Z$.

Observation 2.1. For any $q \in \mathbb{Z}^2 \setminus \{0\}$ we have $|(q + W) \setminus W| > |\mathcal{S}|$.

Although the verification is immediate, calling this fact an observation is deceptive, since W is designed to possess this property. It follows that p is unique, otherwise $|Z| > |W| + |\mathcal{S}| \geqslant |Z_0|$ (since any minimal cover is smaller than the universe), a contradiction.

Lemma 2.2. *Every point $q \in \bar{Z} \setminus Z$ has the same y-coordinate as p.*

Proof. Suppose that there exists $q \in \bar{Z} \setminus Z$ contradicting the statement of the lemma and consider such a q with minimal infection time. Then Z contains at least $|W| - |\mathcal{S}|$ sites on the row of q. Therefore, $|Z| \geqslant 2(|W| - |\mathcal{S}|) > |Z_0|$, a contradiction. □

By monotonicity and Lemma 2.2, we can assume that $p = 0$ (as long as $\langle p, (0,1) \rangle > 0$ we can replace Z by $(Z - (0,1)) \setminus \mathbb{H}_u$ and the problem is invariant under translation by $(\pm 1, 0)$).

Lemma 2.3. *The family $\{S_i \ : \ (i \cdot |\mathcal{S}|, 2) \in Z\}$ is a cover of $\{1, \dots, N\}$.*

Proof. By the minimality of Z and Lemma 2.2, the y-coordinate of any site in Z is 0, 1, or 2. Suppose that $\bar{Z}$ contains $q + W$ for some $q \in \mathbb{Z}^2 \setminus \{0\}$ and take q such that $q + W$ is the first such translate to become infected. By Lemma 2.2 q is of the form $(x, 0)$.

If $|x| \geqslant |\mathcal{S}|^2$, then by Lemma 2.2 the set $Z \setminus W$ contains at least $|\mathcal{S}|$ elements (with y-coordinate 1), therefore $|Z| \geqslant |W| + |\mathcal{S}| \geqslant |Z_0|$, a contradiction.

If $|x| < |\mathcal{S}|^2$, then either some sites in $(q + W) \setminus W$ lying on the x-axis have been infected via rule U_0 or U_1, in which case it is already guaranteed that infinitely many sites become infected during the bootstrap process, or $|Z| \geqslant |Z_0|$ by of Observation 2.1, a contradiction. Thus, removing from Z every site in $Z \setminus W$ with y-coordinate 1 does not prevent the infection of infinitely many sites, which contradicts the minimality of Z.

Hence, such a vector q cannot exist, so that until a rule U_0 or U_1 is used the only possible infections are of the form "$k + (N + j)|\mathcal{S}|^2$ becomes infected via rule $U_{i,j}^k$". Therefore, all sites $(x, 2) \in Z$ are either redundant (which contradicts the minimality of Z) or satisfy $x = i \cdot |\mathcal{S}|$ with $1 \leqslant i \leqslant |\mathcal{S}|$.

Finally, set $I = \{i \ : \ (i \cdot |\mathcal{S}|, 2) \in Z\}$ and assume that

$$J = \{1, \dots, N\} \setminus \bigcup_{i \in I} S_i \neq \varnothing.$$

Then, in order to have $|\bar{Z}| = \infty$, it is necessary (and sufficient) to have a sequence of $N|\mathcal{S}|^2$ consecutive sites in

$$(Z \cap l_u) \cup \{(k + (N + j)|\mathcal{S}|^2, 0) \ : \ i \in I, 1 \leqslant k \leqslant |\mathcal{S}|^2, j \in S_i\}.$$

However, such a sequence is either disjoint from the infections of the form $(k + (N + j)|\mathcal{S}|^2, 0)$, in which case $|Z| \geqslant N|\mathcal{S}|^2$, or disjoint from W. In the latter case the sequence contains at most

$$|Z| - |W| - |I| + (N - |J|) \cdot |\mathcal{S}|^2 \leqslant (|Z_0| - |W|) + (N - 1)|\mathcal{S}|^2 < N|\mathcal{S}|^2$$

infected sites. This contradiction completes the proof. □

It follows from Lemma 2.3 that α_u is indeed equal to $|W|+|M| = |\mathcal{S}|^2 + |\mathcal{S}| + |M|$ as claimed, which completes the proof of Theorem 1.2.

3 Decidability: proof of Theorem 1.4

In this section we provide an algorithm to compute the difficulty of a critical model. Let us stress that it is not optimized and is only meant to prove Theorem 1.4.

Proof of Theorem 1.4. Fix a critical family $\mathcal{U}$. To start, let us note that the stable directions are trivially determined (in polynomial time) and there are at most $|\mathcal{U}|$ isolated stable directions, so it suffices to show that one can compute the difficulty of a given isolated stable direction, since deducing the global difficulty of the model from directional ones is also easy by (1). Let us fix an isolated stable direction u to consider and set

$$D = 2 \cdot \max\left\{ \|x\|_\infty \ :\ x \in \bigcup_{U\in\mathcal{U}} U \right\}, \tag{4}$$

which we shall assume to be sufficiently large throughout the proof.

Recall the notation (3), which we shall use without specifying u, as it will be clear from the context. In order to determine $\alpha(u)$ we will use the following lemmas to bound the size of the set Z in Definition 1.1. The first one is a one-dimensional result that we shall reduce the problem to.

Lemma 3.1. *Let $\mathcal{U}$ be a bootstrap family, let $u \in S^1$ be an isolated stable direction and let $A \subset l_u$. Then the set $\bar{A}$ is either infinite or its maximal distance from A is at most $D^3 \cdot 2^D$.*

Proof. Observe that by stability of u we have $\bar{A} \subset l_u$, so the dynamics (with $\mathbb{H}_u$ fully infected) can be replaced by a one-dimensional bootstrap family acting on l_u, that we identify with $\mathbb{Z}$, so that $\bar{A}$ becomes simply $[A]$. Since u is an isolated stable direction defined by $\mathcal{U}$, distances in l_u are at most D times larger than those in its identification with $\mathbb{Z}$.

Denote $A = \{a_1, \dots, a_n\}$ with $a_1 < \dots < a_n$. Let us denote by P the property that $|[A]| < \infty$, $d(s, A) \leqslant D \cdot 2^{D+1}$ for all $s \in [A]$, $\max[A] - a_n \leqslant D \cdot 2^{D+1} - D$ and $a_1 - \min[A] \leqslant D \cdot 2^{D+1} - D$. Also let $A \neq \varnothing$ be minimal not satisfying P. We aim to prove that $|[A]| = \infty$, so we assume the contrary.

We first note that $|A| > 1$, since, if a single site creates an additional infection, it necessarily creates an infinite arithmetic progression of infections. Assume that there exists $0 < i < n$ and $b \in [A]$ such that $a_{i+1} > b > a_i$ and

$\min(b-a_i, a_{i+1}-b) > D2^{D+1}$. Then by minimality of A both $A' = \{a_1, \dots, a_i\}$ and $A'' = A \setminus A'$ satisfy P. Therefore,

$$\min[A''] - \max[A'] > D \cdot 2^{D+2} - 2(D \cdot 2^{D+1} - D) > D,$$

so that $[A] = [A'] \cup [A'']$, which contradicts the existence of $b \in [A]$.

Assume next that $\max[A] > a_n + D \cdot 2^{D+1} - D$ (the corresponding case for $\min[A]$ is treated identically). Then, by the pigeon-hole principle, there exist $b, c \in \mathbb{Z}$ with $a_n + D < b < c - D < \max[A] - 2D$ such that

$$\varnothing \neq [A] \cap [b, b + D - 1] = ([A] \cap [c, c + D - 1]) - (c - b)$$

(since no infection can cross a region of size D not intersecting $[A]$ to reach $\max[A]$). Therefore, $[A] \cap [b, b + D - 1]$ infects a translate of itself, since the dynamics to the right of $b + D$ is not affected by infections to the left of b, once we fix the state of $b, \dots, b + D - 1$. This is a contradiction with $|[A]| < \infty$, which concludes the proof. □

The next Lemma is an easy application of the covering algorithm of [6].

Lemma 3.2. *Let $\mathcal{U}$ be a critical update family and u be an isolated stable direction. Let $Z \subset \mathbb{H}_{-u}$ be a set of size at most D. Then for every $z \in [Z]$ we have $\langle z, u \rangle \geqslant -\mathcal{O}(D^4)$.*

Proof. We first claim that there exists a set $\mathcal{T} \supset \{u\}$ of three or four stable directions such that for each $v \in \mathcal{T}$ there exists $x \in \mathbb{Z}^2 \cap v\mathbb{R}$ such that $\|x\|_\infty \leqslant D/2$. Indeed, if $-u$ is unstable, it suffices to take the stable directions closest to $-u$ in both semi-circles ending at u and $-u$. These directions satisfy the condition above as they are (semi-)isolated stable for $\mathcal{U}$ and contain the origin, since $\mathcal{U}$ is not supercritical. If, on the contrary, $-u$ is stable, we can pick any (semi-)isolated stable direction in both semi-circles ending at u and $-u$ or, if one of those circles is entirely stable, we take its midpoint. Adding u and $-u$ to those two stable directions, we obtain the desired $\mathcal{T}$.

Then observe that the angle between each two of these directions is $\Omega(1/(D^2))$ (as the determinant of the integer points with those directions is a non-zero integer), so that there is a $\mathcal{T}$-droplet of diameter $\mathcal{O}(D^3)$ containing $\bigcup_{u \in \mathcal{U}} U$. We can then directly apply the covering algorithm of [6] to conclude the proof (using their Lemma 4.6). □

Algorithm. Let us first describe an algorithm to determine $\alpha(u)$ and postpone its analysis. For each integer k from 1 to D we successively perform the following operations to determine if there exists a set Z of size k as in

Definition 1.1. We stop as soon as such a set is found and return the corresponding (minimal) value of k. For each fixed k we start by choosing a set Z_0. The first site is 0 and each new one z is picked within distance $D^{12} \cdot 2^D$ from some of the previous ones and such that $0 \leqslant \langle z' - z, u\rangle = \mathcal{O}(D^4)$ for some z' among the previous ones. There are at most

$$\binom{D^{\mathcal{O}(1)} \cdot 2^D}{D} = \exp(\mathcal{O}(D^2))$$

such choices. For each of them we successively inspect different translations $t \in \mathbb{Z}^2$, such that $0 \leqslant \langle t, u\rangle = \mathcal{O}(D^5)$ and $0 \leqslant \langle t, (-y, x)\rangle < x^2 + y^2$ (where $(-y, x) \in \mathbb{Z}^2$ is such that $(x, y) \in u\mathbb{R}$ and x and y are co-prime), in the (total) order given by $\langle t, u\rangle$ starting from $t = 0$. Finally, fix $Z = Z_0 + t$.

For each Z we run the bootstrap dynamics for $Z \cup \mathbb{H}_u$ until it either stops infecting new sites or infects a site s with $\|s\|_\infty \geqslant D^{14} \cdot 2^D$ and $\langle s, u\rangle = \mathcal{O}(D^5)$. This can be done by checking at each step each site at distance $D^{14} \cdot 2^D + D$ from the origin for each rule and repeating this for 3^D time steps. If the dynamics becomes stationary, we continue, while otherwise we return $|Z|$ for the value of $\alpha(u)$.

Correctness. We now turn to proving that the algorithm does return an output and it is precisely $\alpha(u)$. The first assertion is easy. Indeed, as u is an isolated stable direction, there exists a rule $U \subset \mathbb{H}_u \cup l_u$, so that adding D consecutive sites on l_u to $\mathbb{H}_u$ is enough to infect a half-line of l_u only taking U into account. Thus, we know that $\alpha(u) \leqslant D$ and the algorithm will eventually check such a configuration when $k = D$ and infections will propagate to distance $D^{14} \cdot 2^D$ (and in fact to infinity). Let us then prove that the output is $\alpha(u)$.

Assume that a set $Z = Z_0 + t$ considered by the algorithm is of size $k \leqslant \alpha(u)$ such that $\bar{Z}$ is finite for all previous choices of Z including the current one. We prove by induction on $\langle t, u\rangle$ that the maximal distance between a site from $\bar{Z}$ and Z is at most $D^6 \cdot 2^D \langle t, u\rangle$. Indeed, if $\langle t, u\rangle < 0$, then $Z \subset \mathbb{H}_u$ and there is nothing to prove, since no additional infections take place. Assume the property to hold for all t_j with $j \leqslant i$ in the order given by $\langle t, u\rangle$ and denote $l_j := \{s \in \mathbb{Z}^2, \langle s, u\rangle = \langle t_j, u\rangle\}$ and $Z_j = Z_0 + t_j$ with Z_0 the translate obtained for $t = t_0 = 0$. Observe that by monotonicity for each $0 \leqslant j \leqslant i + 1$ we have that

$$\bar{Z}_{i+1} \cap l_j \subset (\bar{Z}_{i+1-j} \cap l_0) + t_{i+1} - t_{i+1-j},$$

for which the induction hypothesis applies. Thus, we only need to consider $\bar{Z}_{i+1} \cap l_0$. However, by Lemma 3.1, sites there cannot reach distance more than $D + 2^D \cdot D^3$ from

$$(\bar{Z}_{i+1} \cap \mathbb{H}_{-u}) \cup (Z_{i+1} \cap l_0),$$

which is at distance at most $D^6 \cdot 2^D \langle t_i, u\rangle$ from Z_{i+1} itself, by the previous reasoning. This completes the induction (using that $\langle t_{i+1} - t_i, u\rangle$ is independent of i and is at least D^{-2}) and the proof that the algorithm cannot return a value smaller than $\alpha(u)$.

Finally, consider a set $Z \subset \mathbb{Z}^2 \setminus \mathbb{H}_u$ as in Definition 1.1 of size $\alpha(u)$ (and therefore minimal). Note that by minimality and Lemma 3.2, the projection of $Z \cup \{0\}$ on $u\mathbb{R}$ cannot have a gap of length larger than $\mathcal{O}(D^4)$. We also claim that its projection on $(u\mathbb{R})^\perp$ cannot have a gap of length larger than $\mathcal{O}(D^{11} \cdot 2^D)$, which suffices as any set satisfying these conditions is examined by the algorithm and by construction (and the previous reasoning) it will return the first one with $\bar{Z}$ infinite. Indeed, such a gap cannot exist by the reasoning from the previous paragraph applied to each of the two parts of Z separated by the gap. Thus, the output is indeed $\alpha(u)$ and the proof is complete. □

4 Open problems

Let us conclude with a few open questions naturally suggested by the present work. Of course, many more complexity issues arise systematically for hard problems, but let us mention the foremost ones.

Question 1. *Can one find a good approximation of α in polynomial time?*

Question 2. *Are there interesting subfamilies of critical models for which the difficulty is computable in polynomial time?*

Question 3. *In view of Remark 1.5, can one find an algorithm whose complexity depends only on the size of the input $\sum_{U\in\mathcal{U}} |U|$, but not on the size of its entries D? Moreover, is α bounded by a function of the input size and how large is such a function?*

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