Mobile vs. point guards in orthogonal art gallery theorems

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Discrete Geometry Fest, May 15–19 2017, Budapest

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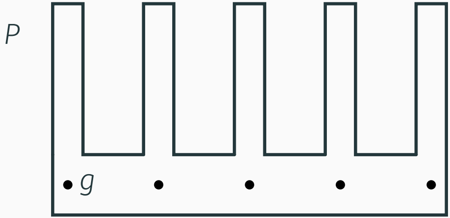
- Art gallery: $P \subset \mathbb{R}^2$, a simple orthogonal polygon
- Point guard: fixed point $g \in P$, has $360°$ line of sight vision
- Objective: place guards in the gallery so that any point in P is seen by at least one of the guards



Typical art gallery theorems

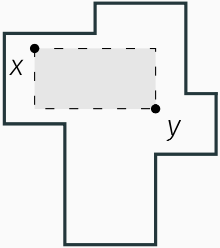
Give (if possible, sharp) bounds on the number of guards required to control the gallery as a function of the number of its vertices.

Theorem (Kahn, Klawe and Kleitman, 1980)

$\lfloor \frac{n}{4} \rfloor$ guards are sometimes necessary and always sufficient to cover the interior of a simple orthogonal polygon of n vertices.

Proof: via convex quadrilateralization.

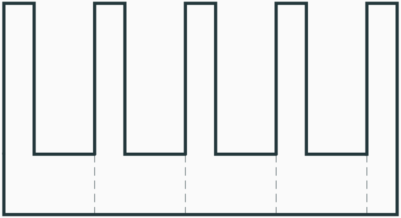
Does the theorem hold if the guards have rectangular vision?



Rectangular vision: two points $x, y \in P$ have r-vision of each other if there is an axis-parallel rectangle inside P, containing x and y.

Theorem (Győri and O'Rourke independently (1983, 1984))

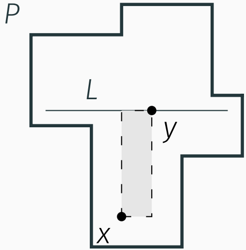
Any n-vertex simple orthogonal polygon can be partitioned into at most $\left\lfloor \frac{n}{4} \right\rfloor$ at most 6-vertex simple orthogonal polygon pieces.



Metatheorem

Every (orthogonal) art gallery theorem has an underlying partition theorem.

A mobile guard is an axis-parallel line segment $L \subset P$ inside the art gallery. The guard sees a point $x \in P$ iff there is a point $y \in L$ such that x is visible from y.



This orthogonal polygon can be covered by one mobile guard

Theorem (Aggarwal, 1984)

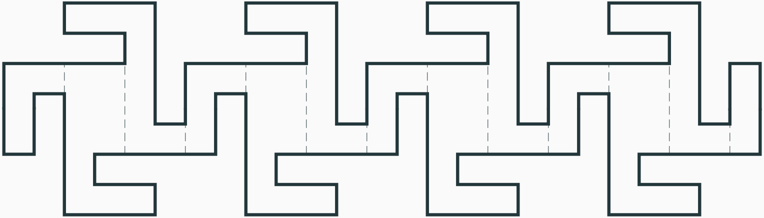
$\left\lfloor \frac{3n+4}{16} \right\rfloor$ mobile guards are sometimes necessary and always sufficient to cover the interior of a simple orthogonal polygon of n vertices.

Two questions of O'Rourke (1987):

- Can crossing patrols be avoided?
- Is it enough that the guards have visibility at the two endpoints of their patrols?

Theorem (Győri, M, 2016)

Any n-vertex simple orthogonal polygon can be partitioned into at most $\lfloor \frac{3n+4}{16} \rfloor$ at most 8-vertex pieces.



Any at most 8-vertex orthogonal polygon can be covered by one mobile guard!

	Point guard	Mobile guard
General polygons	$\left\lfloor \frac{n}{3} \right\rfloor$	$\left\lfloor \frac{n}{4} \right\rfloor$
Orthogonal polygons	$\left\lfloor \frac{n}{4} \right\rfloor$	$\left\lfloor \frac{3n+4}{16} \right\rfloor$

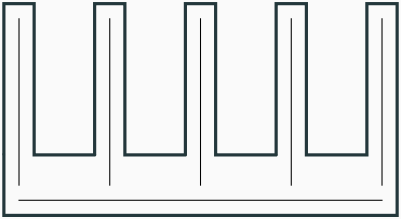
Comparing point guards to mobile guards

	Point guard	Mobile guard
General polygons	$\left\lfloor \frac{n}{3} \right\rfloor$	$\left\lfloor \frac{n}{4} \right\rfloor$
Orthogonal polygons	$\left\lfloor \frac{n}{4} \right\rfloor$	$\left\lfloor \frac{3n+4}{16} \right\rfloor$

$\xrightarrow{3/4}$

Is this $3 : 4$ ratio only an extremal phenomenon?

Maybe...



- One mobile guard can cover a comb, but the minimum number of point guards is equal to the number of teeth.
- Restrict mobile guards to only vertical ones (alternatively, horizontal)!

Definitions

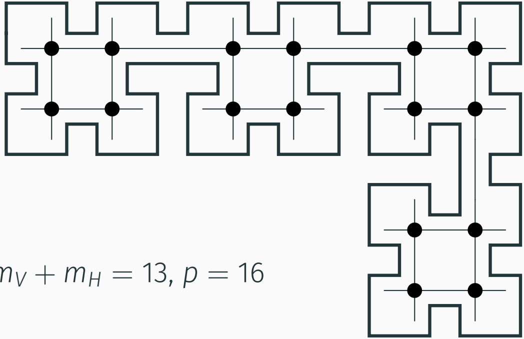
- p: minimum number of point guards required to control P
- m_V: minimum number of mobile guards, whose patrol is a vertical line segment, required to control P
- m_H: minimum number of mobile guards, whose patrol is a horizontal line segment, required to control P

Theorem (Győri, M, 2016)

For any simple orthogonal polygon

$$\frac{m_V + m_H - 1}{p} \geq \frac{3}{4},$$

and this result is sharp.

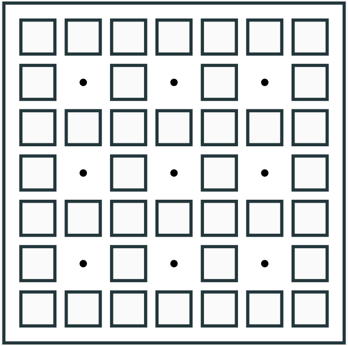


$m_V + m_H = 13,\ p = 16$

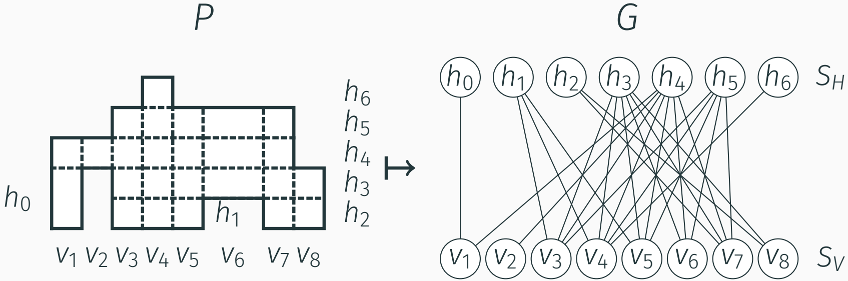
A new block requires 4 more point guards, but only 3 more vertical + horizontal mobile guards.

Simply connectedness is essential

For an orthogonal polygon with orthogonal holes, the ratio of $m_V + m_H$ and p is not bounded: no two of the black dots can be covered by a single point guard.

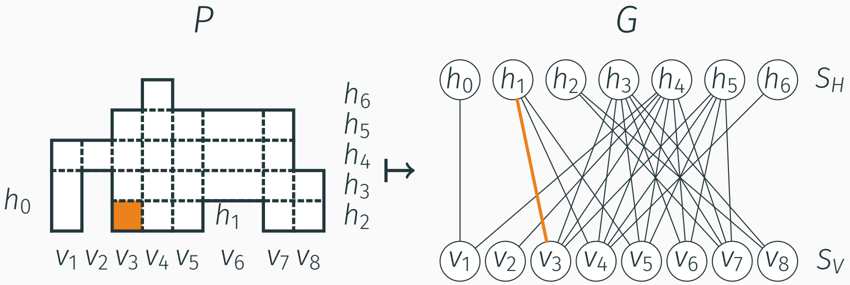


$$m_V + m_H = 4k + 4, \text{ but } p \geq k^2$$



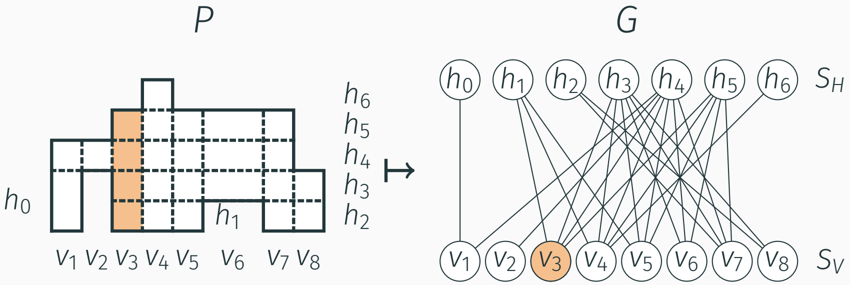
With respect to rectangular vision, it is enough to know the pixels containing the points

Pixelization graph



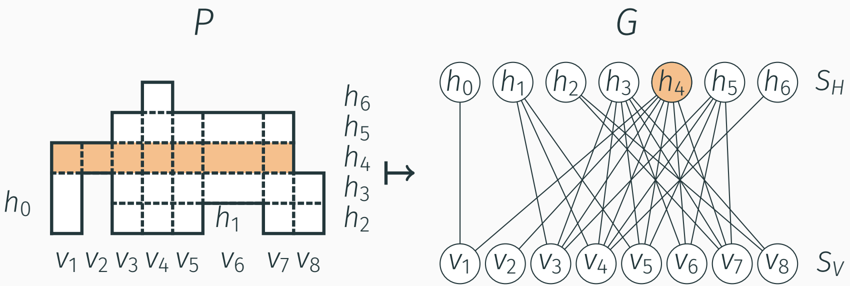
Point guard $\leftrightarrow$ Edge

Pixelization graph

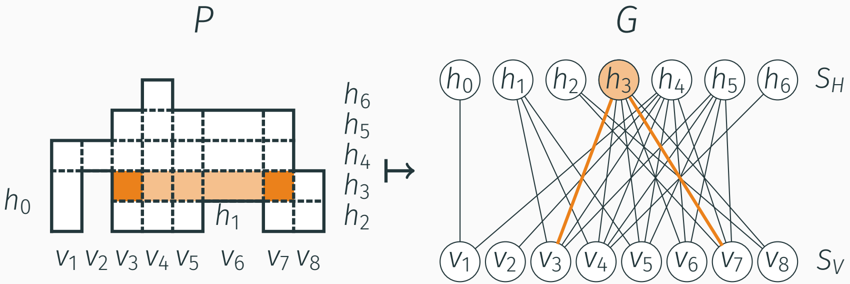


Mobile guard $\leftrightarrow$ Vertex

Pixelization graph

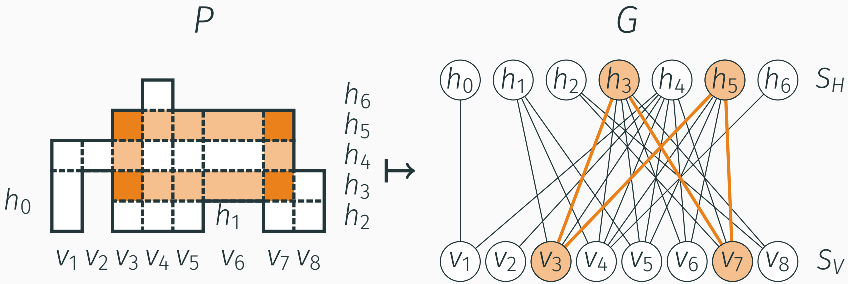


Mobile guard ↔ Vertex

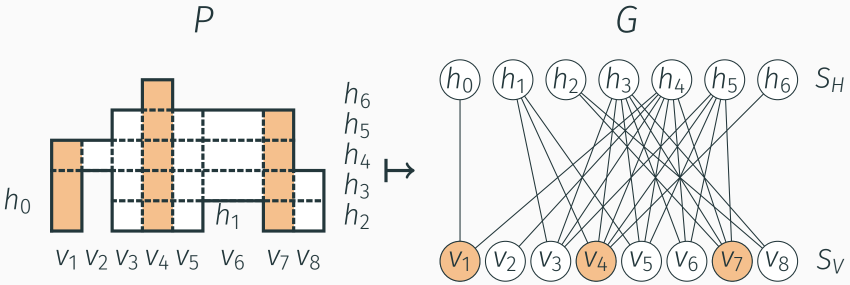


Rectangular vision ($e_1 \cap e_2 \neq \emptyset$)

Pixelization graph



Rectangular vision ($G[e_1 \cup e_2] \cong C_4$)



Vertical mobile guard system $\leftrightarrow M_V \subseteq S_V$ dominating S_H

Orthogonal polygon	Pixelation graph
Mobile guard	Vertex
Point guard	Edge
Simply connected	Chordal bipartite ($\Rightarrow$, but $\not\Leftarrow$)
r-vision of two points	$e_1 \cap e_2 \neq \emptyset$ or $G[e_1 \cup e_2] \cong C_4$
Horiz. mobile guard cover	$M_H \subseteq S_H$ dominating S_V
Covering set of mobile guards	Dominating set

Proof outline — very briefly

- Take $G[M_H \cup M_V]$, it is chordal bipartite as well
- Recursion: first prove the theorem when $G[M_H \cup M_V]$ 2-connected, then connected, and lastly when it has multiple connected components
- The interesting case is when $G[M_H \cup M_V]$ is 2-connected. If we only want to prove a constant of 2, then the proof is 7 pages shorter.

An application of the versus theorem

$$p \leq \frac{4}{3}(m_V + m_H - 1)$$

Theorem (Győri, M, 2016)

For a simple orthogonal polygon given by an ordered list of its vertices, there is a linear time algorithm finding a solution to the minimum size horizontal mobile guard problem.

Trivial observation: both $m_V \leq p$ and $m_H \leq p$.

Corollary

An (8/3)-approximation of the minimum size point guard system for a given orthogonal polygon can be computed in linear time.

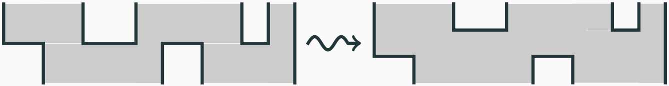
Questions...?!

For interested readers...

- Ervin Győri and M., *Partitioning orthogonal polygons into at most 8-vertex ones, with application to an art gallery theorem* Comput. Geom. 59 (2016), 13–25. `https://arxiv.org/abs/1509.05227`
- Ervin Győri and M., *Mobile vs. point guards*, soon to be submitted (to arXiv as well)

Making the definitions precise

- Degenerate-vision is prohibited
- The vertical and horizontal lines containing a point/mobile guard may not pass through a vertex of the polygon.
- These may be assumed without loss of generality, by using applying the following transformation to the gallery:



Complexity in orthogonal art galleries I.

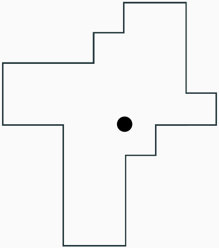
- Worman, Keil (2007): $O\left(n^{17} \cdot \text{polylog}(n)\right)$ algorithm for minimum size point guard system (rectangular vision)
- Lingas, Wasylewicz, Żyliński (2012): linear time 3-approximation for minimum size point guard system (rectangular vision)
- Katz, Morgenstern (2011): finding an minimum size horizontal mobile guard system is polynomial in orthogonal polygons without holes (rectangular vision)

Complexity in orthogonal art galleries II.

- Schuchardt, Hecker (1995): finding a minimum size point guard system is *NP*-hard in simple orthogonal polygons (unrestricted vision)
- Durocher, Mehrabi (2013): optimal mobile guard system is *NP*-hard for orthogonal polygons with holes (rectangular vision)
- Biedl, Chan, Lee, Mehrabi, Montecchiani, Vosoughpour (2016): optimal horizontal mobile guard system is *NP*-hard for orthogonal polygons with holes (rectangular vision)

Theorem (Hoffmann and Kaufmann, 1991)

Any n-vertex orthogonal polygon with holes can be partitioned into at most $\lfloor \frac{n}{4} \rfloor$ at most 16-vertex simple orthogonal star pieces.



A 16-vertex orthogonal star.