# Seating couples and Tic-Tac-Toe

Master's Thesis

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# Contents

**1 Introduction** **1**

**2 Partitioning $\mathbb{Z}_{2p}$ into pairs of prescribed differences** **2**

2.1 Preliminaries . . . . . . . . . . 2

2.2 The main theorem . . . . . . . . . . 5

2.3 Connections to Snevily's conjecture . . . . . . . . . . 8

**3 Applying Theorem 2.4 to the Tic-Tac-Toe game on $\mathbb{Z}^d$** **11**

3.1 Positional Games . . . . . . . . . . 11

3.2 Tic-Tac-Toe on the integer lattice with numerous directions . . . . . . . 12

3.3 Sharpness and auxiliary statements . . . . . . . . . . 17

**4 Simple and small cases** **20**

4.1 Linearly independent winning directions . . . . . . . . . . 20

4.2 Case of $n = 2$ . . . . . . . . . . 21

4.3 Case of $n = 3$ . . . . . . . . . . 21

**5 Conjectures** **25**

**Bibliography** **29**

# Chapter 1

# Introduction

The purpose of this thesis is to achieve new results in Combinatorial Game theory. While writing this thesis, it was also our goal to keep it as self-contained as possible.

In Chapter 2 we prove a variant of the "Seating couples" problem. We later use this theorem in Chapter 3 to prove a sharp result in Combinatorial Game theory (or Tic-Tac-Toe theory). We solve the generalized Tic-Tac-Toe game when there are at most 3 winning directions in Chapter 4. Finally Chapter 5 contains a bouquet of conjectures and unsolved problems, which if proven true would greatly benefit the investigation of the problems that are analyzed in this thesis.

## A note on citation

Statements of theorems and lemmas are always cited unless they are the work of the author (of this thesis). If a proof is not cited but the corresponding statement is, then the proof and the statement are from the same author.

# Chapter 2

# Partitioning $\mathbb{Z}_{2p}$ into pairs of prescribed differences

The need for finding $\frac{p-1}{2}$ disjoint pairs of prescribed differences in $\mathbb{Z}_p$ arises in [MP10]. Recently, this was shown to be possible for any subset of (non-zero) distances independently in [PM09] and [KP12]. Roland Bacher, who posed the problem, also gave a conjecture on generalizing the problem to (odd) finite cyclic groups [Bac08]. In [KP12], the authors discuss a possible way to tackle this generalized problem using the Combinatorial Nullstellensatz [Alo99] and ideas from [DKSS01]. We are going to use a similar approach to prove the conjecture for $\mathbb{Z}_{2p}$, where $p$ is and odd prime.

## 2.1 Preliminaries

**Theorem 2.1** (Combinatorial Nullstellensatz [Alo99]). *Let $\mathbb{F}$ be a field and $f$ a polynomial in $\mathbb{F}[x_1, \ldots, x_n]$. Suppose that $\deg(f) = \sum_{i=1}^{n} t_i$, where $t_i$ are non-negative integers, and that the coefficient of $\prod_{i=1}^{n} x_i^{t_i}$ in $f$ is non-zero. If $S_i \subseteq \mathbb{F}$ and $|S_i| \geq t_i + 1$ for all $i = 1, \ldots, n$, then there exists an $(s_1, \ldots, s_n) \in S_1 \times \ldots \times S_n$, where $f(s_1, \ldots, s_n) \neq 0$.*

*Proof.* ([SM10]) The proof is by induction on $\sum_{i=1}^{n} t_i$. If $\sum_{i=1}^{n} t_i = 0$, then $f \equiv c$, where $c \neq 0$, since it is the main coefficient, so $f$ is nowhere zero on $S_1 \times \ldots \times S_n$.

If $\sum_{i=1}^{n} t_i > 0$, we can assume without loss of generality that $t_1 > 0$. Let $a \in S_1$, and we divide $f$ by $x_1 - a$ as polynomial of $x_1$:

$$f(x_1, \ldots, x_n) = (x_1 - a) \cdot g(x_1, \ldots, x_n) + h(x_2, \ldots, x_n),$$

where $h$ is independent from $x_1$ and in $g$ the coefficient of the monomial $x_1^{t_1-1}x_2^{t_2} \cdots x_n^{t_n}$ is nonzero.

If there exists an $(s_2, \ldots, s_n) \in S_2 \times \ldots \times S_n$ such that $h(s_2, \ldots, s_n) \neq 0$, then $f(a, s_2, \ldots, s_n) = (a - a) \cdot g(a, s_2, \ldots, s_n) + h(s_2, \ldots, s_n) = h(s_2, \ldots, s_n) \neq 0$, so we are done in this case.

On the other hand, if $h$ is everywhere zero on $S_2 \times \ldots \times S_n$, we use induction on $g(x_1, \ldots, x_n)$ and the sets $S_1 \setminus \{a\}, S_2, S_3, \ldots, S_n$ to obtain a vector $(s_1, s_2, \ldots, s_n) \in S_1 \setminus \{a\} \times S_2 \times S_3 \times \ldots \times S_n$ where $g$ does not vanish. Then

$$f(s_1, \ldots, s_n) = (s_1 - a) \cdot g(s_1, \ldots, s_n) + h(s_2, \ldots, s_n) = (s_1 - a) \cdot g(s_1, \ldots, s_n) \neq 0.$$

□

We will also use the Cauchy-Davenport theorem, whose proof is a classical application of the Combinatorial Nullstellensatz, which has numerous extensions, see for example [EK07, KMR11].

**Theorem 2.2** (Cauchy-Davenport)**.** *If $p$ is a prime and $A$, $B$ are two nonempty subsets of $\mathbb{Z}_p$, then*

$$|A + B| \geq \min \{p, |A| + |B| - 1\}.$$

*Proof.* ([Alo99]) If $|A| + |B| \geq p + 1$, then $A \cap \{g - b \mid b \in B\} \neq \emptyset$ by the pigeonhole principle for any $g \in \mathbb{Z}_p$, so there exist an $a \in A$ and $b \in B$ such that $a = g - b$.

If $p \geq |A| + |B|$, suppose for a contradiction that $|A + B| \leq |A| + |B| - 2$. Therefore there exists a subset $C \subseteq \mathbb{Z}_p$ such that $A + B \subseteq C$ and $|C| = |A| + |B| - 2$. Consider the polynomial

$$f(x, y) = \prod_{c \in C}(x + y - c),$$

which vanishes on $A \times B$. The degree of $f$ is $|A| + |B| - 2$, and the coefficient of $x^{|A|-1}y^{|B|-1}$ is $\binom{|A|+|B|-2}{|A|-1}$, which is nonzero, since both $|A| + |B| - 2$ and $|A| - 1$ are less than $p$. The Combinatorial Nullstellensatz provides an $(a, b) \in A \times B$ where $f$ does not vanish, hence we reached a contradiction. □

Unfortunately, the extensions of the Combinatorial Nullstellensatz to commutative unitary rings have additional conditions [KR12, Mez11, KMR11]. Moreover, the main coefficient can easily become 0 if it is the product of non-invertible elements. The easiest way to get around this problem is to embed the additive group (of the ring) into the multiplicative group of a field. For this purpose, we are going to use the following lemma:

**Lemma 2.3** ([DKSS01, p. 3])**.** *Let $k > 1$ be an integer, $c_1, c_2, \ldots, c_k \in \mathbb{C}$ and $S_1, \ldots, S_k \subset \mathbb{C}$, with $|S_i| \geq k$ for all $i$. If the permanent of the Vandermonde-matrix $V(c_1, c_2, \ldots, c_n)$ is non-zero, then there exists $(s_1, s_2, \ldots, s_k) \in S_1 \times S_2 \times \cdots \times S_k$ such that if $i \neq j$ then $s_i \neq s_j$ and $c_i s_i \neq c_j s_j$ $(1 \leq i, j \leq k)$.*

*Proof.* Let us consider the following polynomial:

$$f(x_1, \ldots, x_n) = \prod_{1 \leq i < j \leq k} (x_i - x_j) \cdot \prod_{1 \leq i < j \leq k} (c_i x_i - c_j x_j).$$

It is easy to see that $\deg f = 2\binom{k}{2} = k(k-1)$. Let $t_i = k - 1$, then $|S_i| > t_i$. We want to prove that $f$ takes a non-zero value somewhere on $S_1 \times S_2 \times \cdots S_k$. In order to complete the proof, we only have to apply the Combinatorial Nullstellensatz to $f$, and verify that the coefficient of $\prod_{i=1}^{k} x_i^{k-1}$ in $f$ is non-zero.

By expanding the formula for $f$ we have

$$f(x_1, \ldots, x_k) = \left( \sum_{\sigma} \operatorname{sgn}(\sigma) \prod_{i=1}^{k} x_i^{\sigma(i)} \right) \cdot \left( \sum_{\tau} \operatorname{sgn}(\tau) \prod_{i=1}^{k} (c_i x_i)^{\tau(i)} \right).$$

We only get the desired monomial from the product of the two sums if we choose such a $\sigma$ and $\tau$ that $\sigma(i) + \tau(i) = k - 1$ for all $i$. Let $N(\tau)$ be the number of inversions in $\tau$. Since $\sigma(i) = k - 1 - \tau(i)$, the number of inversion in $\sigma$ is $\binom{k}{2} - N(\tau)$. Therefore we have that $\operatorname{sgn}(\sigma) \cdot \operatorname{sgn}(\tau) = (-1)^{\binom{k}{2}}$, if the product of the monomials corresponding to $\sigma$ and $\tau$ is $\prod_{i=1}^{k} x_i^{k-1}$.

For every $\tau$ permutation we have exactly one suitable $\sigma$, so the coefficient of $\prod_{i=1}^{k} x_i^{k-1}$ in $f$ is

$$(-1)^{\binom{k}{2}} \sum_{\tau} \prod_{i=1}^{k} c_i^{\tau(i)} = (-1)^{\binom{k}{2}} \mathrm{Per} V(c_1, c_2, \ldots, c_k),$$

which is by the assumptions of the lemma non-zero. □

## 2.2 The main theorem

The main result of this chapter is the following theorem. It is closely related to the main theorem of [PM09] (aka the "Seating couples problem").

**Theorem 2.4** (Seating couples)**.** *Let $p$ be a prime. Suppose we are given $p$ elements $d_1, d_2, \ldots, d_p \in \mathbb{Z}_{2p}$, where $d_i$ is odd for all $i = 1, 2, \ldots, p$, and with at most one exception $d_i \neq p$. Then we can partition $\mathbb{Z}_{2p}$ into pairs with differences $d_1, d_2, \ldots, d_p$.*

In fact, we are going to prove a slightly stronger theorem.

**Theorem 2.5.** *Let $p$ be an odd prime. Suppose we are given $p$ elements $d_1, d_2, \ldots, d_p \in \mathbb{Z}_{2p}$, where $d_i$ is odd for all $i = 1, 2, \ldots, p$. Then we can select $2p$ different numbers $a_1, a_2, \ldots, a_p, b_1, b_2, \ldots, b_p \in \mathbb{Z}_{2p}$ such that all $a_i$ are even and all $b_i$ are odd, and $a_i + d_i \equiv b_i \mod 2p$ for all $i = 1, 2, \ldots, p$, if and only if $\sum_{i=1}^{p} d_i \equiv 0 \mod p$.*

*Proof.* It is easy to see that the condition is necessary. The sum of the elements of $\mathbb{Z}_{2p}$ can be counted by summing over all $a_i, b_i$ pairs:

$$\begin{aligned} 0 + 1 + \ldots + 2p - 1 \equiv \sum_{i=1}^{p}(a_i + b_i) \equiv \sum_{i=1}^{p}(2a_i + d_i) &\pmod{2p} \\ \binom{2p}{2} \equiv 4\binom{p}{2} + \sum_{i=1}^{p} d_i &\pmod{2p} \\ p(2p-1) \equiv 2p(p-1) + \sum_{i=1}^{p} d_i &\pmod{2p} \\ p \equiv \sum_{i=1}^{p} d_i &\pmod{2p} \end{aligned}$$

Since each $d_i$ and $p$ is odd, the last equation is equivalent to $\sum_{i=1}^{p} d_i \equiv 0 \mod p$.

Let us now prove that the previous condition is sufficient. Let $a_i$ be a permutation of the even elements of $\mathbb{Z}_{2p}$, and let $b_i = a_i + d_i$, which is an odd number. If $b_i$ are all different elements of $\mathbb{Z}_{2p}$, then $\bigcup_{i=1}^{p}\{a_i, b_i\}$ would be a partitioning of $\mathbb{Z}_{2p}$ into pairs.

We are going to work in the field of complex numbers. We identify $\mathbb{Z}_{2p}$ with a multiplicative subgroup of $\mathbb{C}^*$ which is generated by a primitive root of order $2p$, for example $\varepsilon = e^{i\frac{\pi}{p}}$. We want to apply Lemma 2.3. Let $k = p$, $c_i = \varepsilon^{d_i}$, $S_i \equiv S = \{\varepsilon^{2j} \mid j = 0, 1, \ldots, p-1\}$. If we can prove that $\mathrm{Per}V(\varepsilon^{d_1}, \varepsilon^{d_2}, \ldots, \varepsilon^{d_p})$ is non-zero, the lemma would provide a suitable vector $(\varepsilon^{a_1}, \dots, \varepsilon_p^{a_p}) \in S^p$ for which $\bigcup_{i=1}^p \{\varepsilon^{a_i}\} = S$ and $\bigcup_{i=1}^p \{\varepsilon^{a_i+d_i}\} = \{\varepsilon^{2j+1} \mid j = 0, 1, \dots, p-1\}$. Since $\varepsilon$ is a $2p^{th}$ primitive root, this is equivalent to $\bigcup_{i=1}^p \{a_i, a_i + d_i\}$ being a partition of $\mathbb{Z}_{2p}$.

Now the only thing left for us to prove is that $\mathrm{Per}V(\varepsilon^{d_1}, \varepsilon^{d_2}, \dots, \varepsilon^{d_p})$ is non-zero. Let $Sym(p)$ be the set of all $\sigma : \{1, 2, \dots, p\} \to \{0, 1, \dots, p-1\}$ bijections. Expanding the formula for the permanent, we get

$$\mathrm{Per}V(\varepsilon^{d_1}, \varepsilon^{d_2}, \dots, \varepsilon^{d_p}) = \sum_{\sigma \in Sym(p)} \prod_{i=1}^{p} \varepsilon^{\sigma(i) d_i} = \sum_{j=0}^{2p-1} r_j \varepsilon^j, \tag{2.1}$$

where

$$r_j = |R_j| \quad \text{and} \quad R_j = \left\{ \sigma \in Sym(p) \;\middle|\; \sum_{i=1}^{p} \sigma(i) d_i \equiv j \pmod{2p} \right\}. \tag{2.2}$$

Trivially,

$$\sum_{i=0}^{2p-1} r_i = |Sym(p)| = p! \tag{2.3}$$

We need to differentiate between two very similar cases to continue our analysis.

**Case 1:** $p \equiv 1 \mod 4$.

We claim that $r_{2j+1} = 0$ $(j = 0, \dots, p-1)$. This follows from the fact that for a fixed $\sigma$, its values $\sigma(i)$ take an odd number an even number of times, therefore $\sum_{i=1}^p \sigma(i) d_i$ is even, since each $d_i$ is odd.

Let us analyze the coefficient $r_{2j}$ now. We rewrite the congruence in (2.2):

$$\sum_{i=1}^{p} \sigma(i)(d_i - 1) \equiv 2j - \binom{p}{2} \equiv 2j - p \cdot \frac{p-1}{2} \equiv 2j \pmod{2p},$$

since $\frac{p-1}{2}$ is an even number. Now we can divide the congruence by 2.

$$\sum_{i=1}^{p} \sigma(i) \frac{d_i - 1}{2} \equiv j \pmod{p}. \tag{2.4}$$

If $j \neq 0$, we can multiply (2.4) by any $m$ which is relative prime to $p$ to get a bijection between the elements of $R_j$ and $R_{mj}$:

$$\sum_{i=1}^{p} (m\sigma(i)) \frac{d_i - 1}{2} \equiv mj \pmod{p}.$$

Indeed, $\sigma \longleftrightarrow m\sigma$ is a bijection, because multiplication by $m$ permutes the non-zero elements of $\mathbb{Z}_p$, since it is an invertible element. Thus we have $r_{2j} = r_{2k}$ for all $j, k = 1, 2, \ldots, p-1$.

Furthermore, we claim that $p$ divides $r_0$. What happens if we multiply a $\sigma \in R_0$ by the full cycle permutation? Substituting into (2.4) and using the assumption of the theorem we get

$$\sum_{i=1}^{p}(\sigma(i)+1)\frac{d_i-1}{2} \equiv 0 + \sum_{i=1}^{p}\frac{d_i-1}{2} \equiv \frac{1}{2}\left(\sum_{i=1}^{p} d_i - p\right) \equiv 0 \pmod{p}.$$

Therefore, $R_0$ contains the full orbits of its elements (on the action of the full cycle permutation), which proves our claim.

All in all, we get by re-substituting into (2.1) that

$$\mathrm{Per}V(\varepsilon^{d_1}, \varepsilon^{d_2}, \ldots, \varepsilon^{d_p}) = r_0 + \sum_{j=1}^{p-1} r_2(\varepsilon^2)^j = r_0 - r_2.$$

Suppose for a contradiction, that $r_0 = r_2$. Therefore we have using (2.3) that

$$p \cdot r_0 = r_0 + r_2 + r_4 + \ldots + r_{2p-2} = p!,$$

which implies $r_0 = (p-1)!$. This is a contradiction, since $p$ does not divide $(p-1)!$, so we proved in this case that the permanent cannot be zero.

**Case 2:** $p \equiv 3 \mod 4$.

We claim that $r_{2j} = 0$. For a fixed $\sigma$, its values $\sigma(i)$ take an odd number an odd number of times, therefore $\sum_{i=1}^{p} \sigma(i) d_i$ is odd, since each $d_i$ is odd.

To calculate the coefficient $r_{2j+1}$, we rewrite the congruence in (2.2):

$$\sum_{i=1}^{p}\sigma(i)(d_i-1) \equiv 2j+1-\binom{p}{2} \equiv 2j+1-p\cdot\frac{p-1}{2} \equiv 2j+1-p \pmod{2p},$$

since $\frac{p-1}{2}$ is an odd number. Now we can divide the congruence by 2.

$$\sum_{i=1}^{p}\sigma(i)\frac{d_i-1}{2} \equiv j + \frac{1-p}{2} \pmod{p}. \tag{2.5}$$

From here, the proof is almost the same as in the preceding case, except that the role of $r_0$ will now be played by $r_p$, since the right side of (2.5) is zero for $j = \frac{p-1}{2}$. In other words, the elements in the sum of (2.1) are rotated by 180° compared to the $1^{st}$ case. □

*Remark* 2.6. For $n = p^2$ we have to prove that $r_0 - r_p$ is non-zero.

The main difference between Theorem 2.4 and 2.5 is that in the latter the differences are directed so to speak: if the theorem constructs a pair $\{a_i, b_i\}$ where $a_i - b_i = d_i$, then $a_i$ is even. However, in the preceding theorem this is not assumed, and $a_i$ can very well be odd too.

*Proof of Theorem 2.4.* Again, the case of $p = 2$ can be checked easily. When $p$ is odd, the solution is to allow for both the differences $d_i$ and $-d_i$ between $a_i$ and $b_i$. If we can select $e_i \in \{\pm 1\}$ such that

$$\sum_{i=1}^{p} e_i d_i \equiv 0 \pmod p, \tag{2.6}$$

then we can use Theorem 2.5 to find an appropriate pairing.

Let $T_i = \{d_i, -d_i\}$ for all $i = 1, \ldots, p$. An equivalent formulation of 2.6 is that

$$0 \in T_1 + T_2 + \ldots + T_p \pmod p. \tag{2.7}$$

If $d_i \not\equiv 0 \mod p$ then $|T_i| = 2$, otherwise $|T_i| = 1$. Using the Cauchy-Davenport theorem repeatedly in $\mathbb{Z}_p$ we have that

$$|T_1 + T_2 + \ldots + T_p| \geq \min\left\{p, \sum_{i=1}^{p} |T_i| - (p-1)\right\} \geq$$
$$\geq \min\{p, 2(p-1) + 1 - (p-1)\} \geq p.$$

This obviously implies 2.7, so we are done. □

## 2.3 Connections to Snevily's conjecture

It is of little surprise that Theorem 2.4 and 2.5 have many connections to the recently solved Snevily's conjecture ([Ars11]), since the polynomial used to prove our theorem is identical to the one used in [DKSS01] to prove Snevily's conjecture for finite odd cyclic groups (and a slightly different result on $\mathbb{Z}_{p^\alpha}$). First we prove the following equivalence.

**Claim 2.7.** *Let $n$ be a positive integer, and suppose we are given $k$ elements $d_1, d_2, \ldots, d_k \in \mathbb{Z}$, where $k \le n$ and a subset $A \subseteq \mathbb{Z}_n$ of cardinality $k$. Then the following are equivalent:*

1. *We can find an ordering $a_1, a_2, \ldots, a_k$ of the elements of $A$ such that $a_i + d_i$ are all different in $\mathbb{Z}_n$.*

2. *We can find an ordering $a'_1, a'_2, \ldots, a'_k$ of the elements of $2A \subseteq \mathbb{Z}_{2n}$ such that $a'_i + (2d_i + 1)$ are all different in $\mathbb{Z}_{2n}$.*

*Proof. 2. $\Rightarrow$ 1.* Define $\varphi(2i) = i$ and $\varphi(2i+1) = i$ (where $i = 0, 1, \ldots, p-1$) to get a $\mathbb{Z}_{2n} \to \mathbb{Z}_n$ function. Since $a'_i$ are all even and pairwise different, $a_i = \varphi(a'_i)$ is a permutation of $\mathbb{Z}_p$. Similarly, $a'_i + (2d_i + 1)$ are all odd and pairwise different, so $\varphi(a'_i + (2d_i + 1)) = a_i + d_i \in \mathbb{Z}_p$ are also pairwise different.

*1. $\Rightarrow$ 2.* Let $a'_i = 2a_i$, these are pairwise different. Similarly, $x \mapsto 2x+1$ is an injection of the elements of $\mathbb{Z}_n$ into the odd elements of $\mathbb{Z}_{2n}$, so $2(a_i + d_i) + 1 = a'_i + (2d_i + 1)$ are pairwise different. □

We state the following theorem due to Alon, which proves Snevily's conjecture for cyclic groups of prime order.

**Theorem 2.8** ([Alo00])**.** *Let $p$ be an odd prime, suppose $k < p$, let $A$ be a subset of cardinality $k$ of $\mathbb{Z}_p$ and let $(d_1, \ldots, d_k)$ be a sequence of not necessarily distinct members of $\mathbb{Z}_p$. Then there is an ordering $\{a_1, \ldots, a_k\}$ of the elements of $A$ such that the sums $a_i + d_i$ are pairwise distinct in $\mathbb{Z}_p$.*

Using this theorem we can give a different proof of Theorem 2.5.

*$2^{nd}$ proof of Theorem 2.5.* Let $k = p-1$ and $A = \{1, 2, \ldots, p-1\}$. Apply Theorem 2.8 to the differences $\frac{d_1-1}{2}, \ldots, \frac{d_{p-1}-1}{2} \in \mathbb{Z}_p$. Since $a_i + \frac{d_i-1}{2}$ are pairwise different ($i = 1, \ldots, p-1$), the set $\mathbb{Z}_p \setminus \left\{a_i + \frac{d_i-1}{2},\ i = 1, \ldots, p-1\right\}$ contains exactly one element, and it is equal to

$$\sum_{i=0}^{p-1} i - \sum_{i=1}^{p-1} \left(a_i + \frac{d_i - 1}{2}\right) = -\sum_{i=1}^{p-1} \frac{d_i - 1}{2} = \frac{d_p - 1}{2}$$

since $\sum_{i=1}^{p} d_i \equiv 0 \bmod p$ implies $\sum_{i=1}^{p} \frac{d_i-1}{2} \equiv 0 \bmod p$. If we take $a_p = 0$, we can use Claim 2.7 and we are done. □

*Remark* 2.9. In fact by using Theorem 2.8, we proved a little more: Theorem 2.5 remains true even if we choose any subset $A$ of cardinality $k < p$ of the even elements of $\mathbb{Z}_{2p}$, we can still pair them with odd numbers from $\mathbb{Z}_{2p}$ using any sequence of $k$ odd differences.

*Remark* 2.10. Furthermore, if Theorem 2.5 is true for a number $n$, then Snevily's conjecture is true for $k = n - 1$ in $\mathbb{Z}_n$, even if one of the subsets of cardinality $k$ is a multiset.

# Chapter 3

# Applying Theorem 2.4 to the Tic-Tac-Toe game on $\mathbb{Z}^d$

The theory of combinatorial games is an increasingly popular field in mathematics. It deals with games of complete information where the combinatorial chaos stems from the exponential growth of the number of possible strategies as we increase the size of the game space/table. An exhaustive introduction to the field is [Bec08].

In this chapter we prove a conjecture of Kruczek and Sundberg from [KS08] in a special case using the main result of the previous chapter, Theorem 2.4.

## 3.1 Positional Games

A well-known positional game is Tic-Tac-Toe. The players take turns at putting their marks on the cells of the $3 \times 3$ table until either one of the players occupies a complete line (horizontal, vertical, or diagonal) or no unmarked cells remain. The player who first occupies a complete line is the winner, if there is no winner, the game is a draw.

We can think of the table of the $3 \times 3$ game as a hypergraph $\mathcal{H} = (V, E)$: let $V$ be the 9 cells of the table and $E$ be the winning sets, which are in this case the set of lines of length 3. This hypergraph in itself does not completely define the game Tic-Tac-Toe, and indeed there are many versions of positional games. One such version is the so-called Maker-Breaker (or weak) game: the objective of the first player (Maker) is to occupy a winning set, and the objective of the second player (Breaker) is to keep Maker from doing so. In other words, if Maker cannot occupy a winning set, Breaker is the winner.

The traditional Tic-Tac-Toe can be referred to as a strong positional game. The winner in a strong game is the player who can first occupy a whole winning set (and the game is a draw if there is no winner). The main problem with strong games is that solving them is usually hopeless.

Weak games are still very difficult, but there are techniques to obtain non-trivial strategies, again see [Bec08]. In [KS10], the authors demonstrate a potential based approach to finding winning strategies in the generalized version of Tic-Tac-Toe. This chapter is, however, about constructing pairing strategies for the generalized Tic-Tac-Toe.

A pairing strategy is a (sub)partition of $V$ into pairs. If the other player puts his/her mark on a cell, we mark the cell's pair according to the pairing (if the cell has no pair we mark an arbitrary but yet unmarked cell). By pairing strategy we always mean a winning pairing strategy for Breaker in the weak game.

Another reason to investigate weak games is provided by the following idea: the first player in the strong game can easily adopt Breaker's weak game strategy. Therefore if Breaker can win the weak game, in the strong game any player can force a draw.

A slightly stronger statement worth mentioning is the folklore idea called "Strategy Stealing": if the first player does not have a winning strategy in the strong game, then the first player can at least achieve a draw.

## 3.2 Tic-Tac-Toe on the integer lattice with numerous directions

We define the weak Tic-Tac-Toe game determined by the set of winning directions $S = \{\vec{v}_i \in \mathbb{Z}^d \mid i = 1, \ldots, n\}$, and in the rest of this thesis $n = |S|$. The underlying positional game is given by the hypergraph $\mathcal{H}_S^m = (\mathbb{Z}^d, E_S^m)$, where

$$E_S^m = \left\{ \left\{ \vec{a} + t \cdot \vec{v}_i \mid t = 1, \ldots, m \right\} \;\middle|\; \vec{a} \in \mathbb{Z}^d, \vec{v}_i \in S \right\}$$

is the set of winning sets. If not indicated otherwise, we analyze games where $\vec{v}_i \in S$ are *primitive*: the greatest common divisor of the coordinates of $\vec{v}_i$ is 1. For convenience, we always suppose that $\vec{v}_i \neq \pm\vec{v}_{i'}$ for $i \neq i'$. These two assumptions together imply that any two winning directions are linearly independent.

Kruczek and Sundberg proved in [KS08] that Breaker can win the weak game using a pairing strategy if $m \geq 3n$. They also conjectured that such a strategy exists even if $m \geq 2n+1$ (Conjecture 5.10), and Lemma 3.6 shows that this is sharp. This conjecture was proved to be at least asymptotically true by Mukkamala and Pálvölgyi in [MP10]. Specifically, they proved the following:

**Theorem 3.1** ([MP10]). *If $p$ is a prime and $p = m - 1 \geq 2n + 1$, then in the Maker-Breaker game played on $\mathcal{H}_S^m$, Breaker has a pairing strategy win.*

Unfortunately this theorem does not prove the conjecture of Kruczek and Sundberg for any $n$, since even in the best case scenario we have $m \geq 2n + 2$. However, in the following we are going to use their ideas to prove the following theorem, which proves Conjecture 5.10 in a special case.

**Theorem 3.2.** *Suppose $n = p$ is an odd prime and $m \geq 2n + 1$. If there exists a $\vec{z} \in \{0, 1\}^d$ such that $\vec{v}_i \cdot \vec{z}$ is odd for all $\vec{v}_i \in S$, then in the Maker-Breaker game played on $\mathcal{H}_S^m$, Breaker has a pairing strategy win.*

The rest of this section is a simple modification of the proof of the main theorem of [MP10], but we elaborate on it for the sake of completeness.

*Proof of Theorem 3.2.*

**1. Compactifying the problem**

Let $V_N = \{0, \dots, N - 1\}^d$. First we observe that we only have to prove Theorem 3.2 for $\mathcal{H}_S^m|_{V_N}$ (the sub-hypergraph spanned by $V_N$ in $\mathcal{H}_S^m$) using a compactness argument relying on Kőnig's lemma. A pairing strategy is just a (sub)partition of the vertices of the hypergraph into 2 element subsets. Suppose there is a pairing strategy for Breaker for every positive integer $N$. Let $\mathcal{W}_N$ be the set of all pairing strategies for Breaker on $\mathcal{H}_S^m|_{V_N}$ (define $\mathcal{W}_0 = \{\emptyset\}$). We say that there is an edge between $X \in \mathcal{W}_N$ and $Y \in \mathcal{W}_{N+1}$ if $X$ is the restriction of $Y$ to $V_N$ (we only keep the 2 element subsets).

We can apply Kőnig's lemma to the graph we just defined on $\bigcup_{N=0}^{\infty} \mathcal{W}_N$. The lemma provides an infinitely long path starting from $\mathcal{W}_0$. If we take the union of the vertices of this infinitely long path, we get a partition of $\mathbb{Z}^d$ into pairs. We claim that this is indeed a pairing strategy for $\mathbb{Z}^d$. Suppose Maker can occupy a complete winning set in finite time against this strategy. Then there is a sufficiently large $N$ for which $V_N$ contains all of the points marked by the players. But this cannot be the case, since the restriction of the partition to $V_N$ is a pairing strategy for Breaker.

**2. Mapping $V_N$ into $\mathbb{Z}$**

Let $\vec{r} = (r_1, \ldots, r_d) \in \mathbb{Z}^d$ and let $f_{\vec{r}} = * \cdot \vec{r} : V_N \to \mathbb{Z}$, where $* \cdot *$ is the scalar product. If $r_j > 0$ and $r_{j+1} > N(r_1 + \ldots + r_j)$ for all $j$, then $f_{\vec{r}}$ is injective. The winning directions $\vec{v}_i$ are transformed into $d_i = \vec{v}_i \cdot \vec{r}$. We observe that the winning sets now become arithmetic progressions of length $m$ and difference $d_i$.

Suppose $\vec{u} \in \{0, 1, 2, \ldots, 2p-1\}^d$ and $r_j = u_j + (2pN)^j$. It easily verifiable that $f_{\vec{r}}$ is injective. In addition, we claim that we can choose such a $\vec{u} \in \{0, 1, 2, \ldots, 2p-1\}^d$ that $\gcd(d_i, 2p) = 1$. First we prove that there exists a $\vec{w}$ such that for all $i$

$$\vec{v}_i \cdot \vec{w} \not\equiv \frac{p - \vec{v}_i \cdot \vec{z}}{2} \pmod{p}. \tag{3.1}$$

Suppose we choose $\vec{w}$ uniformly random from $\{0, 1, 2, \ldots, p-1\}^d$, then for a single $i$, $\vec{w}$ does not satisfy incongruence (3.1) with probability $1/p$ (we used the fact that $p$ cannot divide all of the coordinates of $v_i$). Therefore

$$\Pr(\text{for all } i \ \vec{w} \text{ does not satisfy (3.1)}) \leq \sum_{i=1}^{n} \Pr(\vec{w} \text{ does not satisfy (3.1) for } i) \tag{3.2}$$

$$= \sum_{i=1}^{n} \frac{1}{p} = 1,$$

However, we claim that there is strict inequality in (3.2). Equality would only be possible in (3.2) if for all $\vec{w} \in \{0, 1, \ldots, p-1\}^d$ there was a unique $i$ for which $\vec{w}$ satisfies (3.1). But this is not the case, if we let $w_j \equiv -z_j/2 \mod p$ for all $j$, then $\vec{w}$ does not satisfy incongruence (3.1) for any $i$. All in all, we proved that there exists a $\vec{w} \in \{0, 1, \ldots, p-1\}^d$ which satisfies (3.1) for all $i$.

Using that $p - \vec{v}_i \cdot \vec{z}$ is even, we can transform (3.1) to get

$$\vec{v}_i \cdot (2\vec{w} + \vec{z}) \not\equiv p \pmod{2p}, \tag{3.3}$$

and there exists $u_j \in \{0, 1, 2, \ldots, 2p-1\}$ such that $u_j \equiv (2w_j + z_j) \mod 2p$. Moreover

$$\vec{v}_i \cdot \vec{u} \equiv \vec{v}_i \cdot 2\vec{w} + \vec{v}_i \cdot \vec{z} \equiv 0 + 1 \pmod{2},$$

and $\vec{v}_i \cdot \vec{u}$ is not divisible by $p$, so we have $\gcd(\vec{v}_i \cdot \vec{u}, 2p) = 1$. This implies that $\gcd(\vec{v}_i \cdot \vec{r}, 2p) = 1$.

**3. Applying our theorem**

Now we can apply Theorem 2.4 to the differences $d_i = \vec{v}_i \cdot \vec{r}$. Our theorem provides a partition $\bigcup_{i=1}^{p} \{a_i, b_i\} = \mathbb{Z}_{2p}$, where $a_i + d_i \equiv b_i \mod 2p$. This induces a pairing of the elements of $\mathbb{Z}$ in the following way. Suppose $x \in \mathbb{Z}$, if $x \equiv a_i \mod 2p$, we pair $x$ with $x + d_i \in \mathbb{Z}$. Similarly, if $x \equiv b_i \in \mathbb{Z}$, we pair $x$ with $x - d_i \in \mathbb{Z}$. This pairing is well-defined.

To see that this is indeed a pairing strategy for Breaker, consider an arithmetic progression $c_1, \ldots, c_m$ with difference $d_i$. Since $\gcd(d_i, 2p) = 1$, one of $c_1, c_2, \ldots, c_{m-1}$, say $c_j$, is equal to $a_i$ modulo $2p$. Therefore $\{c_j, c_{j+1}\}$ is a pair from our partition of $\mathbb{Z}$, so Maker cannot occupy both points. □

The assumptions of the previous theorem can be relaxed in the following way.

**Theorem 3.3.** *Suppose $n = p$ is an odd prime and $m \geq 2n + 1$. If there exists a $\vec{z} \in \mathbb{Q}$ such that $\vec{v}_i \cdot \vec{z}$ is an odd integer for all $i$, then in the Maker-Breaker game played on $\mathcal{H}$, Breaker has a pairing strategy win.*

*Proof.* The compactness argument can be used again, so we only need to create pairing strategies for Breaker on the bounded parts of the integer lattice. Let $q \in \mathbb{Z}$ be such that $q \cdot \vec{z}$ is an integer vector and let $k \in \mathbb{Z}$ be the largest number for which $\frac{q}{2^k}$ is an integer.

Again, we are going to map $V_N$ into $\mathbb{Z}$ using $f_{\vec{r}}$, thus the winning sets are transformed into arithmetic progressions of length $m$ and difference $d_i = \vec{v}_i \cdot \vec{r}$. We are looking for an $\vec{r}$ of the form $r_j = u_j + \left(2^{k+1}pN\right)^j$, where $u_j \in \left\{0, 1, 2, \ldots, 2^{k+1}p - 1\right\}$, and it is easy to check that $f_{\vec{r}}$ is injective on $V_N$ in this case.

First we find a vector $\vec{w} \in \{0, 1, 2, \ldots, p-1\}^d$ such that

$$\vec{v}_i \cdot \vec{w} \not\equiv \frac{2^k p - \vec{v}_i \cdot (q\vec{z})}{2^{k+1}} \pmod p \tag{3.4}$$

for all $i = 1, 2, \ldots, p$. This can be done just like in the proof of Theorem 3.2, if we let $w_j \equiv -\frac{\vec{v}_i \cdot (qw_j)}{2^{k+1}} \mod p$ for all $j$, then $\vec{w}$ does not satisfy (3.4) for any $i$. Using the assumption we made on $\vec{z}$, we get that $2^k p - \vec{v}_i \cdot (q\vec{z})$ is divisible by $2^{k+1}$, so we can transform (3.4) to get

$$\vec{v}_i \cdot (2^{k+1}\vec{w} + q\vec{z}) \not\equiv 2^k p \pmod{p} \tag{3.5}$$

We can choose a vector $\vec{u} \in \left\{0, 1, 2, \ldots, 2^{k+1}p - 1\right\}^d$ such that

$$u_i \equiv 2^{k+1}w_i + qz_i \pmod{2^{k+1}p}.$$

Moreover, using our assumptions on $\vec{z}$,

$$\vec{v}_i \cdot \vec{u} \equiv \vec{v}_i \cdot \left(2^{k+1}\vec{w}\right) + \vec{v}_i \cdot (q\vec{z}) \equiv \vec{v}_i \cdot (q\vec{z}) \equiv 2^k \pmod{2^{k+1}}.$$

Using this and (3.5), we conclude that $\gcd(\vec{v}_i \cdot \vec{u}, 2^{k+1}p) = 2^k$, and therefore $\gcd(d_i, 2^{k+1}p) = 2^k$.

Now the only thing left for us to do is to find a pairing strategy on $\mathbb{Z}_{2^{k+1}p}$ which blocks any arithmetic progression of length $m$ and difference $d_i$ (in $\mathbb{Z}_{2^{k+1}p}$). This pairing induces a pairing of $\mathbb{Z}$, which can be pulled back to $\mathbb{Z}^d$ since $f_{\vec{r}}$ is injective (there can be unmatched points).

Since $\gcd(d_i, 2^{k+1}p) = 2^k$, the order of $d_i$ in $\mathbb{Z}_{2^{k+1}p}$ is $2p$, and any arithmetic progression of difference $d_i$ in $\mathbb{Z}_{2^{k+1}p}$ stays in the same coset of $2^k\mathbb{Z}_{2^{k+1}p}$. Thus it is enough to partition the elements of the subgroup $2^k\mathbb{Z}_{2^{k+1}p}$ into pairs of differences $d_i$, because the same matching can be used for the rest of the cosets. This problem is equivalent to partitioning the elements of $\mathbb{Z}_{2p}$ using the differences $d_i/2^k$. Since $\gcd(d_i/2^k, 2p) = 1$, there is such a partition of $\mathbb{Z}_{2p}$ according to Theorem 2.4. □

*Remark* 3.4. Let $I = {}_{\mathbb{Z}}(v_1, v_2, \ldots, v_n)$ be the lattice spanned by $\vec{v}_i$. If ${}_{\mathbb{Z}}I \subsetneqq {}_{\mathbb{Z}}\mathbb{Z}^d$, we can think of the game played on $\mathbb{Z}^d$ as multiple Tic-Tac-Toe games played parallel on the cosets of ${}_{\mathbb{Z}}I$: any winning set is completely contained in one of the cosets of ${}_{\mathbb{Z}}I$.

It is worth noting that if we relax the primitiveness condition on the vectors $\vec{v}_i$ to being non-zero, we can derive the following result from the above proof.

**Proposition 3.5.** *Suppose $n = p$ is an odd prime and $m \geq 2n + 1$ and $\vec{v}_i$ are non-zero but not necessarily primitive vectors in $\mathbb{Z}^d$. If there exists a $\vec{z} \in \mathbb{Q}$ such that $\gcd(\vec{v}_i \cdot \vec{z}, 2p) = 1$ for all $i$, then in the Maker-Breaker game played on $\mathcal{H}$, Breaker has a pairing strategy win.*

## 3.3 Sharpness and auxiliary statements

**Lemma 3.6** ([MP10, Section 3]). *If $m \leq 2n$ then Breaker does not have a pairing strategy win on $\mathcal{H}_S^m$.*

*Proof.* It is enough to prove that Breaker cannot win if we restrict the set of points to $V_N = \{0, 1, 2, \ldots, N-1\}^d$ for a large enough $N$. Any set of the form

$$W_i(\vec{x}) = \{\vec{x} + k\vec{v}_i \mid k = 1, 2, \ldots, m-1\} \text{ where } \vec{x} \in V_N \text{ and } \vec{x} + m\vec{v}_i \in V_N,$$

has to contain at least two points paired in the direction of $\vec{v}_i$: otherwise we can extend this set to either $W_i(\vec{x}) \cup \{\vec{x}\}$ or $W_i(\vec{x}) \cup \{\vec{x} + m\vec{v}_i\}$, both of which are winning subsets of $V_N$, but one of them does not contain a pair in the direction of $\vec{v}_i$, so it can be occupied by Maker.

Let us bound from below the maximum number of disjoint $W_i(\vec{x})$ sets we can choose for a fix $i$. If $\|\vec{x}\|_\infty \leq N - m\|\vec{v}_i\|_\infty$, then $W_i(\vec{x})$ can be extended in both directions. Thus we can choose at least $\frac{1}{m-1}(N - m\|\vec{v}_i\|_\infty)^d$ disjoint sets, which can be extended in both directions. These have to contain at least two points paired to each other in the direction of $\vec{v}_i$, thus there are at least $\frac{2}{m-1}(N - m\|\vec{v}_i\|_\infty)^d$ points paired in the direction of $\vec{v}_i$.

Let $C = m \cdot \max_{i=1}^n \|\vec{v}_i\|_\infty$, then we can conclude that there are at least

$$n\frac{2}{m-1}(N-C)^d$$

points paired in the direction of $\vec{v}_1, \ldots, \vec{v}_n$ (a pair can only belong to one winning direction, since any two winning directions are linearly independent). There are only $N^d$ points, so

$$n\frac{2}{m-1}(N-C)^d \leq N^d.$$

Substituting $m \leq 2n$ into the inequation we get

$$\frac{2n}{2n-1} \leq \left(\frac{N}{N-C}\right)^d,$$

or equivalently

$$1 - d\frac{C}{N} \leq \left(1 - \frac{C}{N}\right)^d \leq 1 - \frac{1}{2n},$$

which is a contradiction if $N > 2n \cdot d \cdot C$. $\square$

This lemma states that our theorems are sharp if the vectors are fix and we are looking for a pairing strategy. However, pairing strategies are a small subset of strategies and Combinatorial Game theory offers a variety of ways to construct strategies. Kruczek and Sundberg proved the following asymptotic result using a potential based technique.

**Theorem 3.7** ([KS10]). *Suppose $k \to \infty$ and $d \to \infty$. If $m \geq (1 + o(1))d^2 \log k$, then Breaker has a winning strategy on $\mathcal{H}_S^m$ for any $S \subseteq \{\vec{v} \in \mathbb{Z}^d : \|\vec{v}\|_\infty \leq k\}$.*

Here, the lower bound on $m$ does not directly depend on $n$, but on $d$ and $k$. Informally this theorem states that if there are many dependence relations between the winning directions, Breaker can achieve a win even if it is not possible using a pairing strategy.

Using elementary linear algebra we make two easy statements.

**Corollary 3.8.** *Suppose $n = p$ is a prime and $m \geq 2n + 1$. If $v_i \in \mathbb{Z}^d$ are linearly independent vectors over $\mathbb{F}_2$, then in the Maker-Breaker game played on $\mathcal{H}_S^m$, Breaker has a pairing strategy win.*

*Proof.* Let $M$ be an $n \times d$ matrix that has $v_i$ as its $i^{th}$ row. By our assumptions, the image of $M$ contains $\mathbb{F}_2^n$, in other words there exists a vector $\vec{z} \in \mathbb{F}_2^d$ such that $M\vec{z} = \vec{1}$, so we can apply Theorem 3.2 and we are done. □

If $n = p$, but we cannot apply Theorem 3.2, the following lemma gives us some information about the winning directions that will be useful in Chapter 4 for solving the conjecture of Kruczek and Sundberg when $n = 3$.

**Lemma 3.9.** *Let $M$ be a $n \times d$ matrix over $\mathbb{F}_2$. There exists $\vec{z} \in \mathbb{F}_2^d$ such that $M\vec{z} = \vec{1}$ if and only if no $\vec{y} \in \mathbb{F}_2^n$ such that $\vec{y}M = \vec{0}$, $\vec{y} \cdot \vec{1} = 1$ exists.*

*Proof.* Using the elementary identity $\operatorname{col} M = (\ker M^T)^\perp$, we conclude that $\vec{1} \notin \operatorname{col} M$ if and only if there is a $\vec{y} \in \ker M^T$ which is not perpendicular to $\vec{1}$, that is $\vec{y} \cdot \vec{1} = 1$. □

Using the previous lemma, we can devise a polynomial time algorithm to check if the assumptions of Theorem 3.2 hold. We check using Gauss-elimination if the rows of $M$ are linearly independent. If so, then $\vec{1} \in \operatorname{col} M$. Otherwise, we find a vector $\vec{y}$ such that $\vec{y}M = \vec{0}$. If $\vec{y} \cdot \vec{1} = 1$, the assumptions of Theorem 3.2 do not hold for the rows of $M$. However, if $\vec{y} \cdot \vec{1} = 0$, we can express one of the rows of $M$ as the sum of an odd number of other rows of $M$. Let $M'$ be the matrix we get by deleting this row from $M$. Now there exists a $\vec{y}$ such that $\vec{y}M = 0$ and $\vec{y} \cdot \vec{1} = 1$ if and only if there exists a $\vec{w}$ such that $\vec{w}M' = 0$ and $\vec{w} \cdot \vec{1} = 1$, so we can recurse for $M'$.

# Chapter 4

# Simple and small cases

Although it is not always possible to construct a pairing strategy using the method described in the previous section, in the following we prove that if $m = 2n + 1$, and $n \leq 3$ or the winning vectors are linearly independent then Breaker has a pairing strategy win.

## 4.1 Linearly independent winning directions

**Lemma 4.1.** *Suppose $m \geq 2n+1$ and $v_i \in S \subset \mathbb{Z}^d$ are linearly independent (over $\mathbb{Q}$). Then in the Maker-Breaker game played on $\mathcal{H}_S^m$, Breaker has a pairing strategy win.*

*Proof.* Firstly, it is sufficient to prove the theorem for $m = 2n + 1$. Secondly, we refer to Remark 3.4, that we only have to construct an appropriate pairing on the integer lattice generated by the vectors $\vec{v}_1, \ldots, \vec{v}_n$, since the same pairing can be used on all of its cosets in $\mathbb{Z}^d$. If these vectors are linearly independent, then the lattice generated by them is isomorphic to $\mathbb{Z}^n$ (as a $\mathbb{Z}$-module or as an Abelian group). This follows from the fact that if we take the image of $\vec{v}_i$ to be the $i^{th}$ element ($\vec{e}_i$) of the standard basis in $\mathbb{Q}^n$, we get a linear isomorphism over $\mathbb{Q}$, which is also a $\mathbb{Z}$-module morphism to $\mathbb{Z}^n$, since every element of the integer lattice $_{\mathbb{Z}}(\vec{v}_1, \ldots, \vec{v}_n)$ has its image in $\mathbb{Z}^n$.

Now we only have to construct a suitable pairing for $\mathbb{Z}^n$. Let $\vec{x} = (x_1, \ldots, x_n) \in \mathbb{Z}^n$ and let $i \in \{1, \ldots, n\}$ be the number in the same residue class with $\sum_{j=1}^{n} \lfloor \frac{x_j}{2} \rfloor$ modulo $n$.

If $x_i = 2\lfloor \frac{x_i}{2} \rfloor$, we pair $\vec{x}$ with $\vec{x}+\vec{e}_i$, and if $x_i = 2\lfloor \frac{x_i}{2} \rfloor + 1$, we pair $\vec{x}$ with $\vec{x}-\vec{e}_i$.

Finally, we have to prove that every winning set contains a pair. Take any winning set, for example $\{\vec{z}+r\vec{e}_i \mid r = 0,1,2,\dots,2n\}$. Then there is a $k = 0,1,2,\dots,2n-1$ of the same parity as $x_i$ for which

$$\sum_{j=1}^{n} \left\lfloor \frac{x_j + k\delta_{i,j}}{2} \right\rfloor \equiv i \pmod{n}.$$

It follows from our pairing strategy that $\vec{z}+k\vec{e}_i$ is paired to $\vec{z}+(k+1)\vec{e}_i$, which is also in the winning set. □

*Remark* 4.2. When $n$ is an odd prime, the lemma is also a consequence of Theorem 3.3.

## 4.2 Case of $n = 2$

If $n = 2$, the vectors $\vec{v}_1$ and $\vec{v}_2$ cannot be linearly dependent, as they are both primitive and $\vec{v}_1 \neq \pm\vec{v}_2$. If they are linearly independent, we can refer to Lemma 4.1, but for demonstrative purposes we repeat its proof when $n = 2$. There is a $\mathbb{Z}$-module isomorphism $\varphi$ from ${}_{\mathbb{Z}}(\vec{v}_1,\vec{v}_2)$ to ${}_{\mathbb{Z}}\mathbb{Z}^2$, where $\varphi(\vec{v}_1) = (1,0)$ and $\varphi(\vec{v}_2) = (0,1)$. We can easily block any winning set of size 5 in $\mathbb{Z}^2$ by tiling $\mathbb{Z}^2$ with the following $4 \times 4$ block.

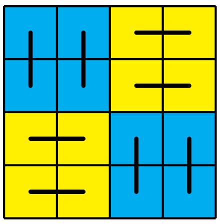


Figure 4.1: This tile defines a periodic pairing strategy on $\mathbb{Z}^2$

The pairing defined on $\mathbb{Z}^2$ can be pulled back to the submodule ${}_{\mathbb{Z}}(\vec{v}_1,\vec{v}_2)$, and we can pair all of its cosets in $\mathbb{Z}^d$ with the same strategy (see Remark 3.4).

## 4.3 Case of $n = 3$

Lemma 4.1 implies that if $\vec{v}_1,\vec{v}_2,\vec{v}_3$ are linearly independent, Breaker has a winning strategy if $m \geq 7$.

If $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are linearly dependent, but there exists a vector $z \in \{0,1\}^d$ such that $\vec{v}_i \cdot \vec{z}$ is odd for $i = 1, 2, 3$, then we can find a pairing strategy using Theorem 3.2.

If there is no such $\vec{z}$, then still there exists a vector $(a, b, c) \in (\mathbb{Z} \setminus \{0\})^3$ such that $a\vec{v}_1 + b\vec{v}_2 + c\vec{v}_3 = \vec{0}$, since no two vectors can be linearly dependent as we saw in the previous case. Without loss of generality we may assume that $\gcd(a, b, c) = 1$ (in other words $(a, b, c)$ is a primitive vector).

We claim that $a, b, c$ are all odd numbers. Let $M$ be the matrix over $\mathbb{F}_2$ formed by taking $\vec{v}_i$ as rows mod 2. Using Lemma 3.9 we conclude that $(1, 1, 1) \in \ker M^T$, since $\vec{v}_i$ are primitive and thus all of them has an odd coordinate. Also, by definition $(a \bmod 2, b \bmod 2, c \bmod 2) \in \ker M^T$. Suppose $a, b, c$ are not all odd, then there is a vector in $\ker M^T$ which contains exactly one odd coordinate, either $(a \bmod 2, b \bmod 2, c \bmod 2)$ or $(a \bmod 2, b \bmod 2, c \bmod 2) + (1, 1, 1)$. Without loss of generality $(1, 0, 0) \in \ker M^T$, but then all of $\vec{v}_1$'s coordinates would be divisible by two, which is a contradiction.

We now want to analyze $(a, b, c) \bmod 6$. By taking either $\vec{v}_1$ or $-\vec{v}_1$, we can change $a$ to $-a$, therefore we may assume that $(a, b, c) \bmod 6$ is in the residue class of one of the elements of $\{1, 3\}^3$. However, it cannot be in the residue class of $(3, 3, 3)$, since $(a, b, c)$ is a primitive vector. It also cannot be in the residue class of $(3, 3, 1)$, since then the coordinates of $\vec{v}_3$ would be divisible by 3, but $\vec{v}_3$ is also primitive. By permuting the vectors $\vec{v}_i$, we may assume that $(a, b, c)$ is in the residue class of either $(1, 1, 1)$ or $(1, 3, 1)$.

Let us take the following (not necessarily surjective) morphism $\varphi : {}_{\mathbb{Z}}\langle \vec{v}_1, \vec{v}_2, \vec{v}_3 \rangle \to {}_{\mathbb{Z}}\mathbb{Z}^2$, where $\varphi(\vec{v}_1) = (c, 0)$ and $\varphi(\vec{v}_2) = (0, c)$. Then we must have that $\varphi(\vec{v}_3) = (-a, -b)$, so this is indeed a $\mathbb{Z}$-module morphism. In addition, $\varphi$ is an isomorphism. Suppose that $\varphi(a'\vec{v}_1 + b'\vec{v}_2 + c'\vec{v}_3) = 0$, that is $a'(c, 0) + b'(0, c) + c'(-a, -b) = 0$, so $a'c = ac'$ and $b'c = bc'$. Therefore $c(a', b', c') = c'(a, b, c)$, and using that $(a, b, c)$ is primitive, there is an $r \in \mathbb{Z}$ such that $c' = rc$, so $(a', b', c') = r(a, b, c)$. This implies that $a'\vec{v}_1 + b'\vec{v}_2 + c'\vec{v}_3 = 0$, which proves that $\varphi$ is indeed an isomorphism.

We now proved that it is enough to find a pairing for the $\mathbb{Z}$-submodule generated by $(c, 0), (0, c), (-a, -b)$ where $(a, b, c) \equiv (1, 1, 1) \bmod 6$ or $(a, b, c) \equiv (1, 1, 3) \bmod 6$. We are going to use the following lemma to construct a pairing strategy in these cases.

If $\vec{v} \in \mathbb{Z}^d$ then by $\gcd \vec{v}$ we mean the greatest common divisor of its coordinates.

**Lemma 4.3.** *Suppose that for all $\vec{v}_i \in S$ we have $\gcd(\gcd \vec{v}_i, 2n) = 1$ and $m \geq 2n+1$. If we can find a partition of $\mathbb{Z}_{2n}^d$ into pairs $\{\vec{x}_i^j, \vec{y}_i^j\}$, where $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, (2n)^{d-1}$, such that $\vec{x}_i^j + \vec{v}_i = \vec{y}_i^j \pmod{2n}$ and $\vec{x}_i^j - \vec{x}_i^{j'}$ is not a multiple of $\vec{v}_i \pmod{2n}$ for $j \neq j'$, then Breaker has a winning pairing strategy on $\mathcal{H}_S^m$.*

*Proof.* Let us denote the submodule of $\mathbb{Z}_{2n}^d$ generated by the single vector $\vec{v}_i$ with $M_i$. We claim that $|M_i| = 2n$. Trivially $|M_i| \leq 2n$. There exists a vector $\vec{w}_i \in \mathbb{Z}^d$ such that $\vec{v}_i \cdot \vec{w}_i = \gcd \vec{v}_i$. Then $\varphi_i(\vec{v}) = \vec{v} \cdot \vec{w}_i$ defines a $\mathbb{Z}_{2n}$-module morphism from $\mathbb{Z}_{2n}^d \to \mathbb{Z}_{2n}$. Thus $\varphi(M_i) = \mathbb{Z}_{2n}$, since the image of $\vec{v}_i$, $\gcd \vec{v}_i$ is an invertible element in $\mathbb{Z}_{2n}$.

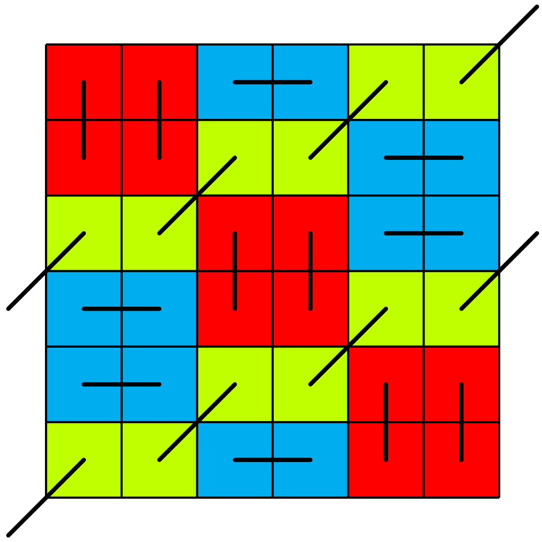
The assumptions on $\vec{x}_i^j$ for a fix $i$ guarantee that we choose an element from all $(2n)^{d-1}$ cosets of $M_i$ in $\mathbb{Z}_{2n}^d$. We pair the elements of $\mathbb{Z}^d$ in the following way. If $\vec{w} \in \mathbb{Z}^d$ is in the equivalence class of $\vec{x}_i^j \in \mathbb{Z}_{2n}^d$, then we pair $\vec{w}$ with $\vec{w} + \vec{v}_i \in \mathbb{Z}^d$. Vica versa, if $\vec{w}$ is in the equivalence class of $\vec{y}_i^j$, then we pair $\vec{w}$ with $\vec{w} - \vec{v}_i \in \mathbb{Z}^d$, so our pairing is well-defined.

Too see that this is indeed a good pairing strategy for Breaker, consider a subset of a winning set $C = \{\vec{c}, \vec{c}+\vec{v}_i, \vec{c}+2\vec{v}_i, \ldots, \vec{c}+(2n-1)\vec{v}_i\} \subset \mathbb{Z}^d$ which does not contain both points of any pair. The image of $C$ through the natural morphism $\mathbb{Z}^d \to \mathbb{Z}_{2n}^d$ is a coset of $M_i$, therefore an element of $C$ is paired to one of the elements in $C \cup \{\vec{c} + (2n)\vec{v}_i\}$. Thus we proved that any winning set contains a pair from the partition of $\mathbb{Z}^d$ defined above. ☐

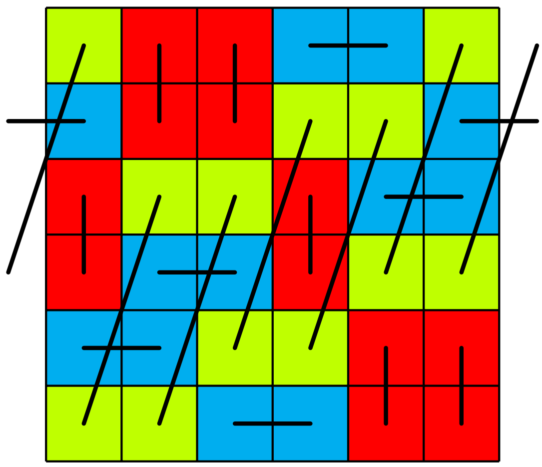
To use the lemma we only have to consider the equivalence class of the vectors $(c, 0), (0, c), (-a, -b)$ in $\mathbb{Z}_6^2$. Therefore we have two cases.

In the first case $a = 1, b = 1, c = 1$, so the winning directions are $\vec{v}_1 = (1, 0), \vec{v}_2 = (0, 1), \vec{v}_3 = (-1, -1)$. Figure 4.2a shows a pairing of $\mathbb{Z}_6^2$ which satisfies the assumptions of the lemma in this case.

In the second case $a = 1, b = 3, c = 1$, so the winning directions are $\vec{v}_1 = (1, 0), \vec{v}_2 = (0, 1), \vec{v}_3 = (-1, -3)$. Figure 4.2b shows a suitable pairing of $\mathbb{Z}_6^2$ for these vectors.



(a) $(1, 0), (0, 1), (-1, -1)$



(b) $(1, 0), (0, 1), (-1, -3)$

Figure 4.2: Pairing strategies on $\mathbb{Z}_6^2$

# Chapter 5

# Conjectures

We state a few conjectures.

**Conjecture 5.1** (Snevily, [Sne99]). *Every $k \times k$ sub-matrix of the addition table of every abelian group of odd order has a latin transversal.*

**Conjecture 5.2** (Snevily, [Sne99]). *Every $k \times k$ sub-matrix of the addition table of $\mathbb{Z}_{2n}$ has a latin transversal, unless it is a translate of a cyclic subgroup of $\mathbb{Z}_{2n}$.*

In fact Conjecture 5.1 was proved just recently by Arsovski [Ars11] after a series of attempts [Alo99, DKSS01]. Unfortunately the proof relies heavily on the fact that a group of odd order is fully representable over a large enough finite field of characteristic 2. A precursor to this idea was used in [DKSS01] to prove Conjecture 5.1 for cyclic groups of odd order. However, it is for this very reason that it is not likely that Arsovski's proof can be easily modified to extend Snevily's conjecture to $\mathbb{Z}_{2n}$.

Bacher, who posted the seating couples problem on an Internet forum [Bac08], conjectured the following.

**Conjecture 5.3** ([Bac08]). *Suppose we are given $n$ elements $d_1, d_2, \ldots, d_n \in \mathbb{Z}_{2n+1}$ that are relative prime to $2n+1$. Then there exists $n$ disjoint pairs in $\mathbb{Z}_{2n+1}$, with differences $d_1, d_2, \ldots, d_n$.*

The conditions are necessary in some sense. For example if $r \in \mathbb{Z}$ is a non-trivial divisor of $2n+1$, then we cannot select $n$ disjoint pairs with difference $r$.

Karasev and Petrov conjectured, that this is the only kind of obstruction that exist for $\mathbb{Z}_{2n}$.

**Conjecture 5.4** ([KP12]). *Suppose we are given $n$ elements $d_1, d_2, \ldots, d_n \in \mathbb{Z}_{2n}$ that are relative prime to $2n$. Then there exists a partition of $\mathbb{Z}_{2n}$ into pairs with differences $d_1, d_2, \ldots, d_n$.*

This conjecture states that a generalization of Theorem 2.4 holds for composite numbers too. Moreover, just as we proved Theorem 2.4 using Theorem 2.5, we can separate Conjecture 5.4 into two sub-problems. The following conjecture is a generalization of Theorem 2.5.

**Conjecture 5.5.** *Let $n$ be a positive integer. Suppose we are given $n$ elements $d_1, d_2, \ldots, d_n \in \mathbb{Z}_{2n}$, where $d_i$ is odd for all $i = 1, 2, \ldots, n$. Then we can select $2n$ different numbers $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n \in \mathbb{Z}_{2n}$ such that all $a_i$ are even and all $b_i$ are odd, and $a_i + d_i = b_i$ for all $i = 1, 2, \ldots, n$, if and only if $\sum_{i=1}^{n} d_i = n \mod 2n$.*

The previous conjecture was verified by a computer program (written by the author) for $n \leq 14$. To deduce Conjecture 5.4 from Conjecture 5.5, we prove the following lemma.

**Lemma 5.6.** *Suppose we are given $n$ elements $d_1, d_2, \ldots, d_n \in \mathbb{Z}$ that are relative primes to $2n$, except $d_1$, which can be any integer. Then there exists a vector $\vec{e} \in \{-1, +1\}^n$ such that $\sum_{i=1}^{n} e_i d_i \equiv n \mod 2n$.*

*Proof.* Let $S_k = \{\sum_{i=1}^{k} e_i d_i \ : \ \vec{e} \in \{\pm 1\}^n\}$. First we prove an easy statement.

**Claim 5.7.** *Suppose $S \subseteq 2\mathbb{Z}_{2n}$ or $S \subseteq 1 + 2\mathbb{Z}_{2n}$ and $d \in \mathbb{Z}$ such that $\gcd(d, 2n) = 1$. Then*

$$|S \cup (S + 2d)| \geq \min\{n, |S| + 1\}.$$

*Proof.* We may assume that $|S| < n$ (otherwise we are done). Suppose for a contradiction that $|S| = |S \cup (S + 2d)|$. This implies that $S = S + 2d$ and that $S$ contains a coset of the subgroup $D = \{2kd : \ k \in \mathbb{Z}\} \leq \mathbb{Z}_{2n}$. Using that $\gcd(2d, 2n) = 2$, we have that $|D| = n$ which implies $|S| \geq n$, a contradiction. □

We prove by induction on $k$ that $|S_k| \geq k$ for any $k \leq n$. Trivially, $|S_1| \geq 1$. Suppose that $|S_{k-1}| \geq k-1$. Let $S = S_{k-1} - d_k$, then we can use the claim to infer that

$$|S_k| = |(S_{k-1} - d_k) \cup (S_{k-1} + d_k)| = |S \cup (S + 2d_k)| \geq k,$$

since every element of $S_{k-1} - d_k$ is the sum of exactly $k$ odd numbers, so $S_{k-1} - d_k \subseteq k + 2\mathbb{Z}_{2n}$ satisfies the assumptions of the claim.

We now know that $|S_n| = n$ and that $S_n \subseteq n + 2\mathbb{Z}_{2n}$, which together imply that $n \in S_n$, the claim of the lemma. □

In Section 2.3 we gave another proof of Theorem 2.5 by reducing the problem to a version of Snevily's conjecture. Using Claim 2.7 we can write Conjecture 5.5 in the following equivalent form. This form is also a generalization of a special case of both Theorem 2.8 and Conjecture 5.2.

**Conjecture 5.8** (Special case of the Kézdy-Snevily conjecture [KS02])**.** *Let $n$ be a positive integer, $A = \{1, 2, 3, \dots, n-1\} \subset \mathbb{Z}_n$ and let $(d_1, \dots, d_{n-1})$ be a sequence of not necessarily distinct members of $\mathbb{Z}_n$. Then there is a permutation $\{a_1, \dots, a_{n-1}\}$ of the elements of $A$ such that the sums $a_i + d_i$ are pairwise distinct in $\mathbb{Z}_n$.*

If Conjecture 5.5 is proved, it would imply the existence of a pairing strategy for many Tic-Tac-Toe games. For example, the following conjecture would be true (which is the analogue of Proposition 3.5).

**Conjecture 5.9.** *Suppose $m \geq 2n+1$ and $\vec{v}_i$ are non-zero but not necessarily primitive vectors. If there exists a $\vec{z} \in \mathbb{Q}$ such that $\vec{v}_i \cdot \vec{z} \in \mathbb{Z}$ and $\gcd(\vec{v}_i \cdot \vec{z}, 2n) = 1$ for all $i$, then in the Maker-Breaker game played on $\mathcal{H}_S^m$, Breaker has a pairing strategy win.*

Kruczek and Sundberg conjectured the following statement.

**Conjecture 5.10** ([KS08])**.** *If $m \geq 2n+1$, Breaker has a pairing strategy win on $\mathcal{H}_S^m$.*

An even stronger conjecture is due to Mukkamala and Pálvölgyi.

**Conjecture 5.11** ([MP10])**.** *Suppose we are given $n$ primitive vectors $\vec{v}_i$ in $\mathbb{Z}_{2n}^d$. We can always find a partition of $\mathbb{Z}_{2n}^d$ into pairs $\{\vec{x}_i^j, \vec{y}_i^j\}$, where $i = 1, 2, \dots, n$ and $j = 1, 2, \dots, (2n)^{d-1}$, such that $\vec{x}_i^j + \vec{v}_i = \vec{y}_i^j$ and $\vec{x}_i^j - \vec{x}_i^{j'}$ is not a multiple of $\vec{v}_i$ for $j \neq j'$.*

This is also a generalization of Conjecture 5.4. In Theorem 2.4 we proved this for $d = 1$ when $n$ is a prime. Lemma 4.3 shows that Conjecture 5.11 is indeed stronger than Conjecture 5.10.

One may ask whether it is possible that Conjecture 5.10 holds, but Conjecture 5.11 does not. Another similar question is whether every pairing strategy for Breaker when $m = 2n + 1$ can be constructed by Lemma 4.3 using an appropriate pairing strategy on $\mathbb{Z}_{2n}^d$ or not. We suspect that the answer to the first question is no.

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