Extremal solutions to some art gallery and terminal-pairability problems

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PhD defense,
17 November 2017, Budapest

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Art gallery problems	Terminal-pairability problem
abstract models of challenges that appear in the world	
geometric algorithms	edge-connectivity, network flow
image processing, VLSI design	routing traffic in networks
NP-hard, unknown to be polynomial time, or $O(n^{17})$	

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The metamethod: the extremal-type approach

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2. In each group, find (up to a constant) sharp bounds on the optimal solution of the worst case in the group

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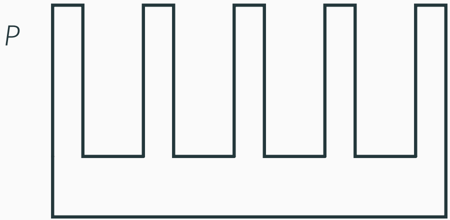
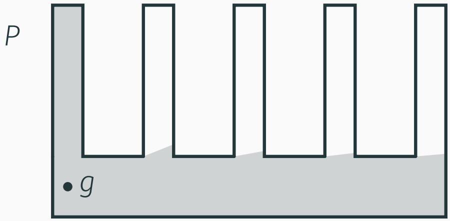
2. In each group, find (up to a constant) sharp bounds on the optimal solution of the worst case in the group
3. For an instance of the problem in the group, a solution achieving the above bound is usually a good approximation of the optimum
4. Moreover, such a solution can often be constructed efficiently (in polynomial time)

Results in art gallery problems

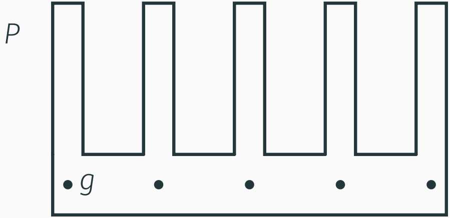
- Art gallery: $P \subset \mathbb{R}^2$, a simple orthogonal polygon



- Art gallery: $P \subset \mathbb{R}^2$, a simple orthogonal polygon
- Point guard: fixed point $g \in P$, has 360° line of sight vision



- Art gallery: $P \subset \mathbb{R}^2$, a simple orthogonal polygon
- Point guard: fixed point $g \in P$, has 360° line of sight vision
- Objective: place guards in the gallery so that any point in P is seen by at least one of the guards



Give (if possible, sharp) bounds on the number of guards required to control the gallery as a function of the number of its vertices.

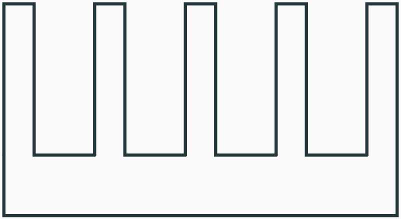
Theorem (Kahn, Klawe and Kleitman, 1980)

$\lfloor \frac{n}{4} \rfloor$ guards are sometimes necessary and always sufficient to cover the interior of a simple orthogonal polygon of n vertices.

Proof: via convex quadrilateralization.

Theorem (Győri and O'Rourke independently, around 1984)

Every orthogonal polygon of n vertices can be partitioned into $\lfloor \frac{n}{4} \rfloor$ orthogonal polygons of at most 6 vertices.



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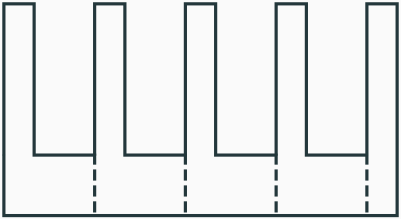

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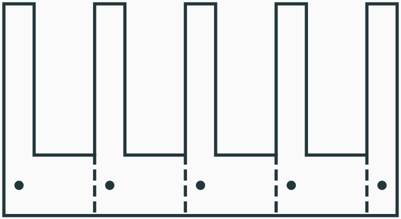
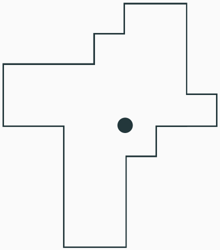


Theorem (Hoffmann, 1990)

Any n-vertex orthogonal polygon with holes can be partitioned into at most $\left\lfloor \frac{n}{4} \right\rfloor$ at most 16-vertex simple orthogonal star pieces.

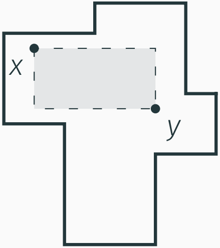


A 16-vertex orthogonal star.

Metatheorem

Every (orthogonal) art gallery theorem has an underlying partition theorem.

Rectangular vision: two points $x, y \in P$ have r-vision of each other if there is an axis-parallel rectangle inside P, containing x and y.



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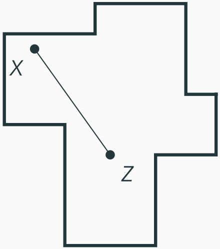
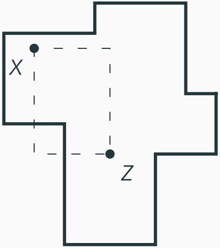


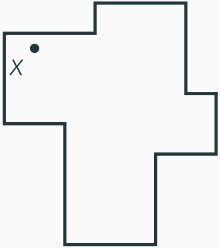
x and z have unrestricted vision of each other

Rectangular vision: two points $x, y \in P$ have r-vision of each other if there is an axis-parallel rectangle inside P, containing x and y.



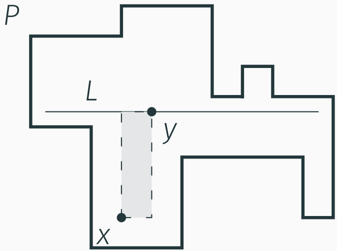
x and z do not have rectangular vision of each other

Rectangular vision: two points $x, y \in P$ have r-vision of each other if there is an axis-parallel rectangle inside P, containing x and y.



The extremal bound is the same with rectangular vision

A mobile guard is an axis-parallel line segment $L \subset P$ inside the art gallery. The guard sees a point $x \in P$ iff there is a point $y \in L$ such that x is visible from y.



This orthogonal polygon can be covered by one mobile guard

Theorem (Aggarwal, 1984)

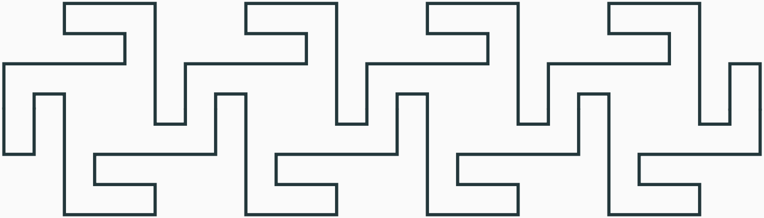
$\left\lfloor \frac{3n+4}{16} \right\rfloor$ mobile guards are sometimes necessary and always sufficient to cover the interior of a simple orthogonal polygon of n vertices.

Two questions of O’Rourke (1987):

- Can crossing patrols be avoided?
- Is it enough that the guards have visibility at the two endpoints of their patrols?

Theorem (Győri, M, 2016)

Any n-vertex simple orthogonal polygon can be partitioned into at most $\lfloor \frac{3n+4}{16} \rfloor$ at most 8-vertex pieces.



Partitioning orthogonal polygons

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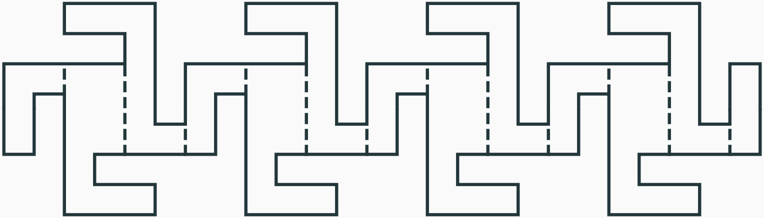

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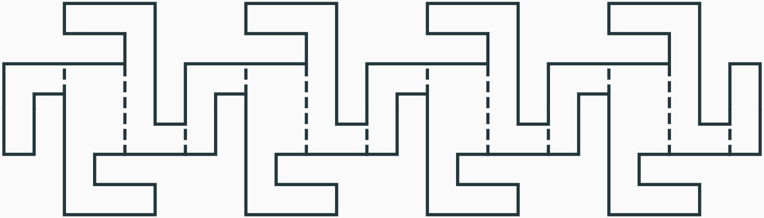

Any at most 8-vertex orthogonal polygon can be covered by one mobile guard!

- The complexity of the minimum size mobile guard system is unknown, probably NP-hard.
- The previous partitioning theorem can be turned into a linear time algorithm.

Comparing point guards to mobile guards

	Point guard	Mobile guard
General polygons	$\left\lfloor \frac{n}{3} \right\rfloor$	$\left\lfloor \frac{n}{4} \right\rfloor$
Orthogonal polygons	$\left\lfloor \frac{n}{4} \right\rfloor$	$\left\lfloor \frac{3n+4}{16} \right\rfloor$

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$\xrightarrow{3/4}$

Is this $3:4$ ratio only an extremal phenomenon?

Definitions

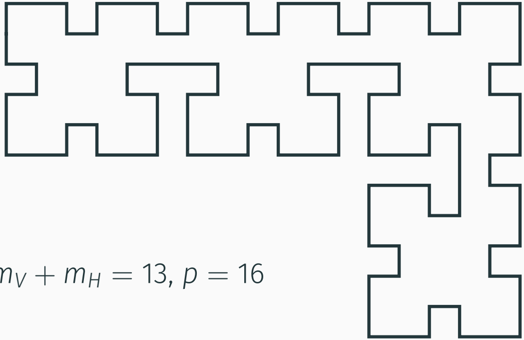
- p: minimum number of point guards required to control P
- m_V: minimum number of mobile guards, whose patrol is a vertical line segment, required to control P
- m_H: minimum number of mobile guards, whose patrol is a horizontal line segment, required to control P

Theorem (Győri, M, 2016)

For any simple orthogonal polygon

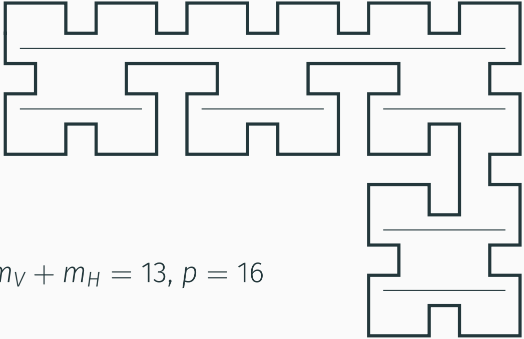
$$\frac{m_V + m_H - 1}{p} \geq \frac{3}{4},$$

and this result is sharp.



$m_V + m_H = 13$, $p = 16$

A new block requires 4 more point guards, but only 3 more vertical + horizontal mobile guards.



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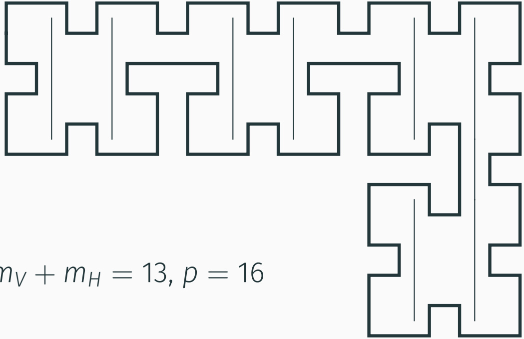
Sharpness



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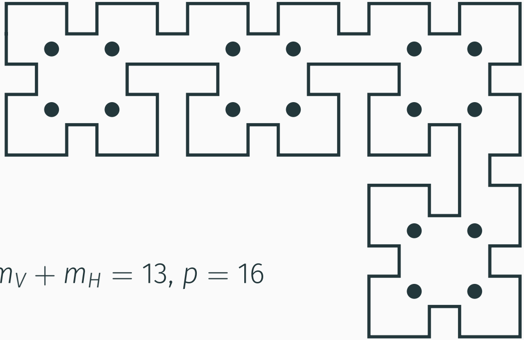

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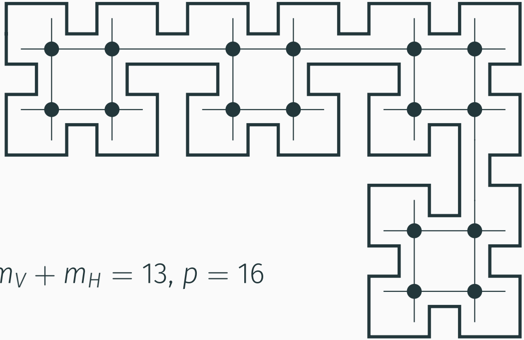

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Approximating the optimal point guard system

Theorem (Győri, M, 2016)

The minimum size horizontal mobile guard system can be computed in linear time.

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Corollary

An $\frac{8}{3}$-approximation of the minimum size of a point guard system of an orthogonal polygon can be computed in linear time.

Theorem (Győri, M, 2016)

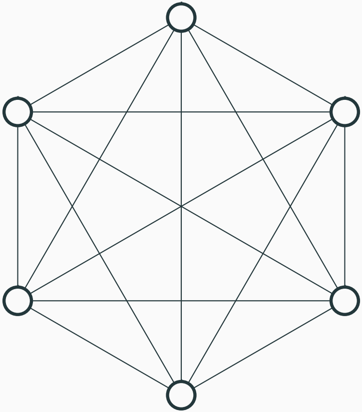
The minimum size horizontal mobile guard system can be computed in linear time.

Corollary

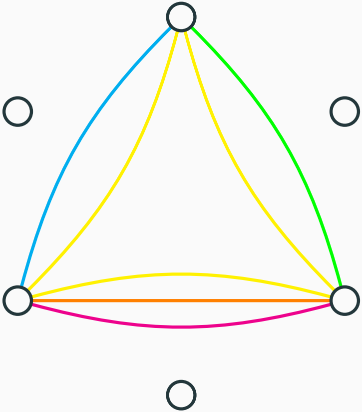
An $\frac{8}{3}$-approximation of the minimum size of a point guard system of an orthogonal polygon can be computed in linear time.

Optimal point guard problem: generally NP-hard. However, Worman and Keil (2007) showed that it can be computed in $\tilde{O}(n^{17})$ for orthogonal polygons and rectangular vision.

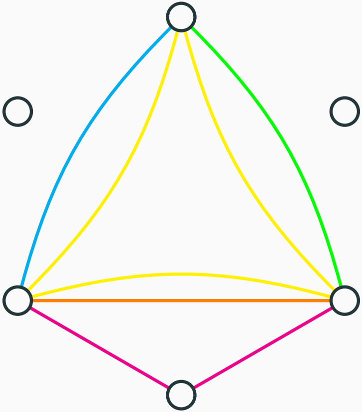
Results in terminal-pairability



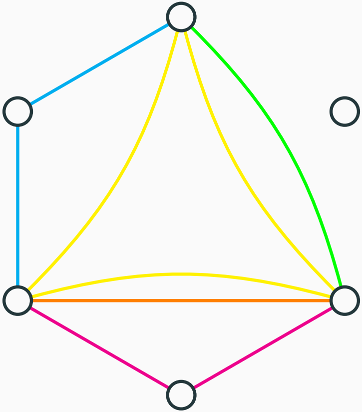
Base graph



Demand graph



Resolving multiplicities



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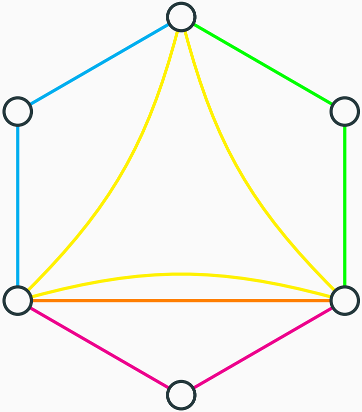

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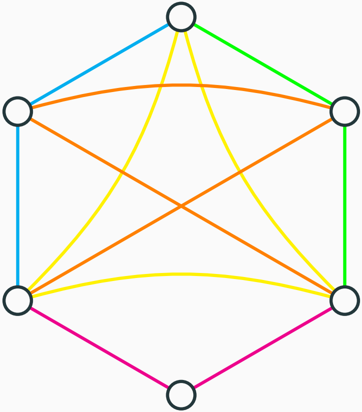
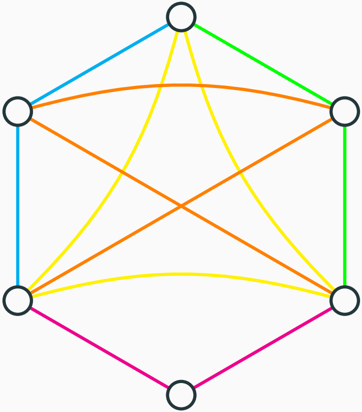


Solution/Realization

Edge-disjoint paths problem



EDP is NP-hard (Karp, 1972), even for complete base graphs

- Solving every instance of the edge-disjoint paths problem separately is hopeless
- Let's test a single base graph against a set of demand graphs charaterized by a degree restriction.
- Motivation: given network switches with a fixed number of ports, build larger switches from them as components

Problem (Csaba, Ralph J. Faudree, András Gyárfás, Jenő Lehel, and Schelp, 1992)

What is the highest number q for which any demand graph on n vertices and maximum degree q is realizable in K_n?

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Theorem (Csaba, Ralph J. Faudree, András Gyárfás, Jenő Lehel, and Schelp, 1992)

$$\frac{n}{7.5} \leq q \leq \frac{n}{2}.$$

Theorem (Győri, M, Mészáros, 2016)

Any demand graph D on n-vertices with $\Delta(D) \leq 2\lfloor \frac{n}{6} \rfloor - 4$ is realizable in K_n.

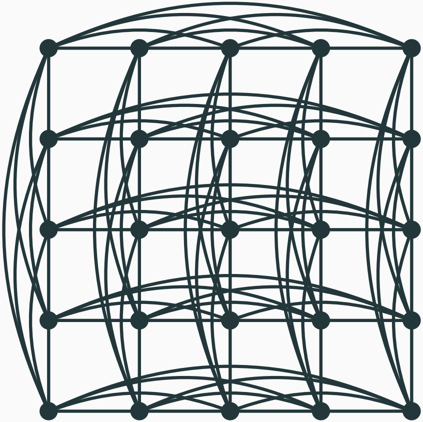
Theorem (Kosowski, 2008)

Let D be a demand graph on the vertex set of K_n. There is an $O(mn \log n)$ time algorithm which gives a 3.75-approximation solution to the MaxEDP problem in K_n.

Theorem (M, 2017)

Let D be a demand graph on the vertex set of K_n. There is an $O(mn \log n + n^3)$ time algorithm which gives a $(3 + O(1/n))$-approximation solution to the MaxEDP problem in K_n.

Complete grid graph, $d = 2$



$$K_5 \square K_5 = K_5^2$$

Theorem (Győri, M, Mészáros, 2016)

Let $G = K_t^d$ and let $D = (V(D), E(D))$ be a demand graph with $V(D) = V(K_t^d)$ and $\Delta(D) \leq 2\lfloor \frac{t}{12} \rfloor - 2$. Then D can be realized in G.

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G is **path-pairable**, if any matching of its vertices can be realized in G.

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G is **path-pairable**, if any matching of its vertices can be realized in G.

Corollary

If $t \geq 24$, K_t^d is path-pairable.

Suppose any matching can be realized in G and $\Delta(G) \leq \Delta$.
What is the maximum of $N = |V(G)|$?

Theorem (Ralph J. Faudree, András Gyárfás, and Jenő Lehel, 1999)

$$N \leq 2\Delta^{\Delta}.$$

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Theorem (Mészáros, 2015)

$$\Delta^2 \leq N$$

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Theorem (Mészáros, 2015)

$$\Delta^2 \leq N$$

$$24^{\Delta/23} \leq N \leq 2\Delta^{\Delta}.$$

Thank you for your attention!

Accepted publications I

- Ervin Győri, Tamás Róbert Mezei, (2016). “Partitioning orthogonal polygons into $\leq$ 8-vertex pieces, with application to an art gallery theorem”. In: *Comput. Geom.* 59, pp. 13–25.
- Ervin Győri, Tamás Róbert Mezei, Gábor Mészáros, (2016). “Terminal-Pairability in Complete Graphs”. In: *J. Combin. Math. Combin. Comput.* Accepted for publication.
- Lucas Colucci, Péter L Erdős, Ervin Győri, Tamás Róbert Mezei, (2017a). “Terminal-Pairability in Complete Bipartite Graphs”. In: *Discrete Appl. Math.* Accepted for publication.

- Dóra Dedinszki, Flóra Szeri, Eszter Kozák, Viola Pomozi, Natália Tőkési, Tamás Róbert Mezei, Kinga Merczel, Emmanuel Letavernier, Ellie Tang, Olivier Le Saux, Tamás Arányi, Koen Wetering, András Váradi, (2017). "Oral administration of pyrophosphate inhibits connective tissue calcification". In: *EMBO Molecular Medicine* 9.11, pp. 1463–1470.
- Ervin Győri, Tamás Róbert Mezei, Gábor Mészáros, (2017). "Note on terminal-pairability in complete grid graphs". In: *Discrete Math.* 340.5, pp. 988–990.

Submitted publications

- Lucas Colucci, Péter L Erdős, Ervin Győri, Tamás Róbert Mezei, (2017b). "Terminal-Pairability in Complete Bipartite Graphs with Non-Bipartite Demands". In: *submitted to Theoret. Comput. Sci.*
- Ervin Győri, Tamás Róbert Mezei, (2017). "Mobile vs. point guards". In: *submitted to Discrete Comput. Geom.*

I am very grateful to...

- My supervisor, *Ervin* (who in any situation has a relevant story about a well-known mathematician),
- *Gábor Mészáros*, coauthor of the papers about terminal-pairability,
- My parents,
- My partner, *Eszter*,
- My alma mater, *CEU*

for their continuous support during my studies and the writing of my thesis.