

Terminal-Pairability in Complete Bipartite Graphs with Non-Bipartite Demands

Edge-disjoint paths in complete bipartite graphs

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ABSTRACT. We investigate the terminal-pairability problem in the case when the base graph is a complete bipartite graph, and the demand graph is a (not necessarily bipartite) multigraph on the same vertex set. In computer science, this problem is known as the edge-disjoint paths problem. We improve the lower bound on the maximum value of $\Delta(D)$ which still guarantees that the demand graph D has a realization in $K_{n,n}$. We also solve the extremal problem on the number of edges, i.e., we determine the maximum number of edges which guarantees that a demand graph is realizable in $K_{n,n}$.

Keywords: edge-disjoint paths; terminal-pairability; complete bipartite graph

1. INTRODUCTION

The *terminal-pairability* problem has been introduced in [7]. The basic question is as follows: Let G be a simple graph — the *base graph*, and let D be a loopless multigraph with the same vertex set $V(D) = V(G)$ — the *demand graph*. Can we find a path $P(e)$ for every edge $e \in E(D)$ such that $P(e)$ joins the end-vertices of e and these paths are pairwise edge-disjoint? If there is such a collection of paths then we say that D is *realizable* in G . The collection of paths is called a *realization* of D in G .

This problem has several names in the literature depending on motivation and background. In the terminal-pairability context, sufficient conditions (which guarantee the existence of a realization) and their extremum are sought after. In computer science, where the problem is referred to as the *edge-disjoint paths* problem (EDP problem for short), the complexity of constructing the set of

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edge-disjoint paths is studied. In the following few paragraphs we take a short detour to survey the previous results about the complexity of the EDP problem.

The decision version of EDP was first shown to be NP-complete by Even, Itai, and Shamir [11]. Robertson and Seymour [31] proved that for a fixed number of paths the problem is solvable in polynomial time, and the running time was later improved by [23] (these results are about vertex-disjoint paths, but by moving to the line graph of G , edge-disjoint paths become vertex-disjoint). However, if the number of required paths is part of the input then the problem is NP-complete even for complete (see [25]) and series-parallel graphs (see [30]). The problem is NP-hard even if $G + D$ is Eulerian and D consists of at most three set of parallel edges, as shown by Vygen [35]. If no restrictions are made on G , then the problem is NP-hard for one set of parallel edges which should be mapped to edge-disjoint paths of length exactly 3, see [3].

The edge-disjoint paths problem has many practical applications in telecommunications, VLSI design, network science, see [15], for example. Therefore several random and deterministic methods have been developed to solve this problem for special graph classes and cases when the number of demand edges is not too high. From a long series of papers we single out the paper of Alon and Capalbo [2], where the interested reader can also find a good survey of the earlier developments. Theorem 1.2 of [2] is the best possible (in magnitude) on very strong d -regular expanders: no vertex can occur more than $d/3$ demand pairs, furthermore there are at most $\Theta\left(\frac{nd \log d}{\log n}\right)$ demand pairs.

In [19], a solution is described for the special case of the complete graph with the same magnitude of demand pairs, but the constant there is much better than in [2]. An advantage of Alon and Capalbo's solution is that it is based on an online algorithm. It receives the demand edges one-by-one, and designs the paths immediately, without any information on the still forthcoming demand edges.

Theorem 1.2 of [2] does not apply for bipartite base graphs; instead, Theorem 1.1 of [2] can be used. For the special case of complete bipartite base graphs, our approach increases the number of possible demand edges asymptotically by a factor of $\log(2n)$ from $\frac{n^2}{75 \log(2n)}$ to $\frac{n^2}{4}$ (though it should be mentioned that the upper bound on the degrees in the demand graph is $n/3$ in Theorem 1.1 of [2]).

The problem is also very closely related to the integer multicommodity flow problems and the theory of graph immersions, each with their own terminologies. In this paper from now on we use the terminology of terminal-pairability, as other papers [7, 13, 12, 14, 26, 18, 27, 28, 24, 29, 19] about sufficient conditions do.

The terminal-pairability problem arose as a theoretical framework for the practical problem of constructing high throughput packet switching networks. The problem was originally studied by Csaba, Faudree, Gyarfas, Lehel, and Schelp [7]. Their research served as a substrate for further theoretical studies by Gyarfas and Schelp [18] and Kubicka, Kubicki, and Lehel [27].

The NP-hardness of the problem studied means that it is hopeless to give a condition which is both necessary and sufficient for D to be realizable in G .

Instead, we group the instances of the problem according to the value of a parameter which corresponds to the complexity of it: the maximum degree, or the number of edges of D . Given a fixed value of one of these parameters, we are able to give relatively tight conditions which guarantee the existence of a realization. Moreover, for an instance of the problem satisfying these conditions a solution can be constructed in polynomial time.

This paper is the latest piece in a series of papers about terminal-pairability [19, 20, 6]. Our previous paper in the series [6] also deals with terminal-pairability in complete bipartite base graphs, but with the not very natural restriction that D is bipartite with respect to the same vertex classes as the base graph. The novelty of this paper is that almost the same conditions are sufficient even if the bipartiteness condition on D is omitted.

For an edge $e \in E(D)$ with end-vertices x and y , we define the *lifting of e to a vertex $z \in V(D)$* , as an operation which transforms D by deleting e and adding a new edge joining the vertices x and z and another new one joining z and y ; if $z = x$ or $z = y$, the operation does not do anything. We stress that we do not use any information about G to perform a lifting and that the graph obtained using a lifting operation is still a demand graph D' . Notice that D has a realization if D' has a realization. Throughout the paper, the demand graphs will be denoted by D and its (indexed) derivatives.

Note that the terminal-pairability problem defined by G and D is solvable if and only if there exists a series of liftings, which, applied successively to D , results in a (simple!) subgraph of G . This subgraph is called a *realization* of D in G . The edge-disjoint paths can be recovered by assigning pairwise different labels to the edges of D , and performing the series of liftings so that new edges inherit the label of the edge they replace. Clearly, edges sharing the same label form a walk between the endpoints of the demand edge of the same label in D , and so there is also a such path.

Several attempts have been made to improve the general result for the case of complete base graphs. One of them is due to Kosowski [25], and an even better extremal bound is proven in [19].

In our previous paper [6] on terminal-pairability in complete bipartite graphs, we restricted ourselves to demand graphs that are bipartite with respect to the color classes of the base graph.

In this paper we explore the terminal-pairability problem when G is a complete symmetric bipartite graph, and the demand graph is assumed to be a loopless multigraph. Our approach is an extremal one, and we do not consider the computational complexity of finding the solution (although our proofs lend themselves to be turned into randomized polynomial time algorithms).

Let $A = \{a_1, \dots, a_n\}$ and $B = \{b_1, \dots, b_n\}$. We define $K_{n,n}$ as the graph whose vertex set is $A \cup B$, and its edge set is the set of all unordered A, B pairs. For a (multi)graph H , whose vertex set is $A \cup B$, let $H[A, B]$ be the bipartite subgraph of H induced by A and B as the two color classes. For a (multi-)graph H , let $\Delta(H)$ be the maximum degree in H , and let $e(H)$ be the number of its edges.

A motivation behind taking $K_{n,n}$ as the base graph is that from a bundle of parallel edges between the two color classes, each edge (except at most one) is mapped to a path of at least 3 edges. If the base graph is a complete graph, multiple edges only need to be mapped to paths of length at least 2, so studying terminal-pairability in complete bipartite graphs is a logical next step for this reason, too.

The observation of the previous paragraph and the pigeonhole principle implies that a demand graph containing $\lceil \frac{n}{3} \rceil + 1$ copies of the edge $\{a_i, b_i\}$ for each $i = 1, \dots, n$ is not realizable in $K_{n,n}$. Therefore, the extremal upper bound on the maximum degree of a demand graph realizable in $K_{n,n}$ is at most $\lceil \frac{n}{3} \rceil$.

Theorem 1. *Let D be a demand graph such that $V(D) = V(K_{n,n})$ (but D is not necessarily bipartite). If*

$$\Delta(D) \leq (1 - o(1)) \cdot \frac{n}{4}$$

as $n \rightarrow \infty$, then D is realizable in $K_{n,n}$.

Such sufficient conditions are studied in the theory *immersions* as well. It is a relatively new and quickly growing subject of graph theory. In short, a loopless (multi)graph D has an immersion in a graph G if there exists a mapping of $V(D)$ into $V(G)$ so that D is realizable in G with respect to this vertex-map. The foundations have been laid down by Robertson and Seymour [32], Abu-Khzam and Langston [1], and Fellows and Langston [15, 16]. Recently, Collins and Heenehan [5], Devos, Dvořák, Fox, McDonald, Mohar, and Scheide [8], and Dvořák and Yepremyan [10] studied the problem of finding sufficient conditions on a simple graph so that it contains an immersion of K_n . Theorem 1 (and even more so, the main result in [19]) studies the converse of this problem: what is a sufficient condition on a loopless multigraph D so that it has an immersion into $K_{n,n}$ (respectively, K_n). In comparison, Theorem 2 in [6] is not invariant on the permutation of $V(D)$, but the technical novelty of this paper (compared to [6]) allows us to reformulate Theorem 1 in the language of immersions.

Corollary. *If H is a loopless multigraph on at most $2n$ vertices with maximum degree at most $(1 - o(1)) \cdot \frac{n}{4}$, then there is an immersion of H in $K_{n,n}$.*

Although Theorem 1 is not likely to be sharp, it is definitely a $\frac{4}{3}$ -approximation of the extremal bound. If a significant amount of the edges of D are inside the two color classes, Theorem 2 permits even higher degrees in D .

Theorem 2. *Let D be a demand graph (which is not necessarily bipartite), such that $V(D) = V(K_{n,n})$. If*

$$\Delta(D) \leq (1 - o(1)) \cdot \left(\frac{2n}{7} - \frac{3}{7} \cdot \frac{e(D[A, B])}{n} \right)$$

as $n \rightarrow \infty$, then D is realizable in $K_{n,n}$.

Additionally, we prove a sharp bound on the maximum number of edges in a realizable demand graph:

Theorem 3. *Let $n \geq 2$ and D be a demand graph with the base graph $K_{n,n}$. If D has at most $2n - 3$ edges and $\Delta(D) \leq n$, then D is realizable in $K_{n,n}$.*

The assumption $\Delta(D) \leq n$ is trivially necessary: at any given vertex, there can be at most n edge-disjoint paths that terminate there. The result is sharp, as shown by the demand graph on $2n - 2$ edges consisting of two bundles of $n - 1$ edges, where one of the bundles joins an arbitrary pair of vertices in A , while the other bundle joins a pair in B .

The following NP-hardness result is probably well-known, but we have not been able to find a reference for it. For completeness' sake, we include a short reduction.

Claim 4. *The terminal-pairability problem in $K_{n,n}$ is NP-hard.*

Proof. Take an instance of the edge-disjoint paths problem in K_n ; let D be the demand graph (with vertex set $\{a_1, \dots, a_n\}$) of which we need to decide whether it is realizable in K_n or not. Take $G = K_{\binom{n}{2}, \binom{n}{2}}$, such that its vertex classes are $\{a_1, \dots, a_{\binom{n}{2}}\}$ and $\{b_1, \dots, b_{\binom{n}{2}}\}$. Let the edge set of D' , a demand graph on the vertex set of G , be

$$E(D') = E(D) \cup E(G) \setminus \left\{ (a_i, b_j), (b_j, a_k) : 1 \leq i < k \leq n, j = i - 1 + \sum_{l=0}^{k-1} l \right\}.$$

Obviously, if D has a realization in K_n then we may lift an edge $\{a_i, a_k\}$ of the realization of D to the vertex b_j such that $j = i - 1 + \sum_{l=0}^{k-1} l$. If D' has a realization in G then it also has such a realization where each edge

$$e \in E(G) \setminus \left\{ (a_i, b_j), (b_j, a_k) : 1 \leq i < k \leq n, j = i - 1 + \sum_{l=0}^{k-1} l \right\}$$

is mapped to a path of one edge, i.e., itself: this can clearly be done, if e is not in the realization of D' ; if it is, we can simply modify a path $P(f) \ni e$ by replacing $P(f)$ with $P(f) \cup P(e) \setminus \{e\}$ (some cycles may have to be pruned). Such a solution trivially corresponds to a realization of D in K_n . The described reduction is polynomial. \square

2. PROOFS OF THE DEGREE VERSIONS (THEOREM 1 AND 2)

Before we proceed to prove the theorems, we state several definitions and four well-known results about edge colorings of multigraphs.

Let H be a loopless multigraph. Recall, that the *chromatic index* of H (also known as the *edge chromatic number*), denoted by $\chi'(H)$, is the minimum k such that there is a proper k -coloring of the edges of H . An *equitable edge coloring* of H is a proper coloring of the edges $E(H)$ such that the sizes of the color classes differ by at most one. The *list chromatic index* of H (also known as the *list edge chromatic number*), denoted by $\text{ch}'(H)$, is the smallest integer k such that if for each edge of H there is a list of k different colors given, then there exists a proper coloring of the edges of H where each edge gets its color from its list.

The *maximum multiplicity* $\mu(H)$ is the maximum number of edges joining the same pair of vertices in H . The number of edges joining a vertex $x \in V(H)$ to a subset $A \subseteq V(H)$ of vertices is denoted by $e_H(x, A)$. The *set of neighbors* of x in H is denoted by $N_H(x)$. For other notation the reader is referred to [9].

Claim 5. *If H is a multigraph and $\chi'(H) \leq k$ for some integer k , then there is an equitable edge coloring of H with exactly k colors.*

Proof. Let $c : E(H) \rightarrow \{1, 2, \dots, k\}$ be a proper edge coloring of H . Suppose there are two colors i and j for which $|c^{-1}(i)| \geq |c^{-1}(j)| + 2$. The connected components of $H_{i,j} = c^{-1}(i) \cup c^{-1}(j)$ are cycles (where two parallel edges are regarded as a 2-cycle) and paths. In any cycle of $H_{i,j}$, the number of edges of color i is equal to the number of edges of color j , therefore $H_{i,j}$ must contain a path component of odd length, with one more edge of color i than of color j . By switching the two colors in this path, the sum $\sum_{i=1}^k |c^{-1}(i)|^2$ decreases, and we end up with a coloring which is still proper. Thus, if we cannot repeat this procedure anymore, c must be an equitable coloring, as desired. \square

We will use the following well-known results about the edge colorings of multigraphs.

Theorem 6 (Vizing [34]). *For any multigraph H , its chromatic index*

$$\chi'(H) \leq \Delta(H) + \mu(H).$$

Theorem 7 (Shannon [33]). *For any multigraph H , its chromatic index*

$$\chi'(H) \leq \frac{3}{2}\Delta(H).$$

Theorem 8 (Kahn [21]). *For any multigraph H , its list chromatic index*

$$\text{ch}'(H) \leq (1 + o(1))\chi'(H).$$

Claim 9. *If D is a demand graph on the vertex set $V(K_{n,n})$ and $\Delta(D) \leq n/3$, then there exists a proper edge $2\lfloor n/2 \rfloor$ -coloring of $D[A, B] \cup D[B]$, which is an equitable $2\lfloor n/2 \rfloor$ -coloring on both $D[A, B]$ and $D[B]$.*

Proof. Observe that (by Theorem 7)

$$\begin{aligned} \chi'(D[B]) &\leq \frac{3}{2}\Delta(D) \leq \frac{1}{2}n, \\ \chi'(D[A, B]) &\leq \frac{3}{2}\Delta(D) \leq \frac{1}{2}n. \end{aligned}$$

By Claim 5, there is a partition of $E(D[B])$ into $\lfloor n/2 \rfloor$ matchings of size $\lfloor e(D[B])/n \rfloor$ and $\lceil e(D[B])/n \rceil$, say $M_1, \dots, M_{\lfloor n/2 \rfloor}$, so that $|M_i| \geq |M_j|$ for $i < j$. Similarly, there is a partition of $E(D[A, B])$ into $\lfloor n/2 \rfloor$ matchings of size $\lfloor e(D[A, B])/n \rfloor$ and $\lceil e(D[A, B])/n \rceil$, say $N_1, \dots, N_{\lfloor n/2 \rfloor}$, so that $|N_i| \leq |N_j|$ for $i < j$. It is sufficient to prove now that for all $i = 1, \dots, \lfloor n/2 \rfloor$, there exists an equitable 2-coloring of $M_i \cup N_i$.

Observe, that $M_i \cup N_i$ is the vertex disjoint union of some edges, and paths composed of two or three edges that alternate between elements of M_i and N_i .

The paths of two and three edges contain one edge of M_i exactly. Color the M_i edge of half of the path components of length three with color 1, and color the other half of the M_i edges in the paths of length three with color 2. Do so with the path components of length two as well. The colors of edges of N_i intersecting M_i edges are now determined. This partial 2-coloring is proper, and almost equitable. As the yet uncolored edges of $M_i \cup N_i$ are vertex disjoint, this partial coloring can be extended to an almost equitable 2-coloring of $M_i \cup N_i$. \square

Proof of Theorem 1. By adding edges, if necessary, we may assume that D is regular. Clearly, $e(D[A]) = e(D[B])$, $e(D) = e(D[A]) + e(D[A, B]) + e(D[B])$, and $e(D) = n \cdot \Delta(D)$.

The proof consists of three steps. In the first step, we resolve the high multiplicity edges of $D[A]$, while leaving $D[A, B] \cup D[B]$ untouched. In the second step, we lift the edges of $D[B]$ to A , and resolve the multiplicities of $D[A, B]$. In the third step, we lift the edges induced by A to B , while preserving a simpleness of the bipartite subgraph induced by A and B , thus we end up with a graph which is a realization of D .

By Theorem 7, $\chi'(D[A]) \leq n$, so Claim 5 implies the existence of an equitable edge n -coloring c_1 of $D[A]$. We construct D' from D by lifting the elements of $c_1^{-1}(i)$ to a_i for all $i = 1, \dots, n$. As c_1 is a proper coloring, $\mu(D'[A]) \leq 2$. For any $a \in A$ and $b \in B$, we have the following estimates:

$$\begin{aligned} e_{D'}(a, A) &\leq e_D(a, A) + 2 \cdot \lceil e(D[A])/n \rceil, \\ e_{D'}(a, B) &= e_D(a, B), \quad e_{D'}(b, A) = e_D(b, A), \quad e_{D'}(b, B) = e_D(b, B). \end{aligned}$$

For the second step, we use Claim 9 to take a proper edge n -coloring c_2 of $D[A, B] \cup D[B]$, which is an equitable n - or $(n-1)$ -coloring if restricted to both $D[A, B]$ and $D[B]$. We get D'' from D' by lifting the elements of $c_2^{-1}(i)$ to a_i for all $i = 1, \dots, n$. As c_2 is a proper edge coloring, $D''[A, B]$ is simple, and $\mu(D''[A]) \leq \mu(D'[A]) + 2 \leq 4$. For any $a \in A$ and $b \in B$, we have the following estimates:

$$\begin{aligned} e_{D''}(a, A) &\leq e_{D'}(a, A) + e_{D'}(a, B) + \lceil e(D[A, B])/n \rceil \\ e_{D''}(a, B) &\leq \lceil e(D[A, B])/(n-1) \rceil + 2 \cdot \lceil e(D[B])/(n-1) \rceil \\ e_{D''}(b, A) &= \Delta(D), \quad e_{D''}(b, B) = 0. \end{aligned}$$

To each edge $e \in E(D''[A])$ with end vertices a_i and a_j , we associate a list $L(e)$ of vertices of B , to which we can lift e to without creating multiple edges:

$$L(e) = B \setminus (N_{D''}(a_i) \cup N_{D''}(a_j)),$$

whose size is bounded from below

$$\begin{aligned} |L(e)| &\geq n - e_{D''}(a_i, B) - e_{D''}(a_j, B) \geq \\ &\geq n - 2 \lceil e(D[A, B])/(n-1) \rceil - 4 \cdot \lceil e(D[B])/(n-1) \rceil. \end{aligned}$$

By Vizing's theorem (Theorem 6),

$$\begin{aligned} \chi'(D''[A]) &\leq \Delta(D''[A]) + \mu(D''[A]) \leq \\ &\leq \max_{a \in A} (e_{D'}(a, A) + e_{D'}(a, B) + \lceil e(D[A, B])/(n-1) \rceil) + 4 \leq \\ &\leq \Delta(D) + 2 \cdot \lceil e(D[A])/n \rceil + \lceil e(D[A, B])/(n-1) \rceil + 4. \end{aligned}$$

Furthermore, by Kahn's theorem (Theorem 8), $\text{ch}'(D''[A]) \leq (1+o(1))\chi'(D''[A])$. We have $\text{ch}'(D''[A]) \leq |L(e)|$ for each edge e in $E(D''[A])$, if

$$\begin{aligned} (1+o(1))(\Delta(D) + 2 \cdot \lceil e(D[A])/n \rceil + \lceil e(D[A, B])/(n-1) \rceil) &\leq \\ &\leq n - 2\lceil e(D[A, B])/(n-1) \rceil - 4 \cdot \lceil e(D[B])/(n-1) \rceil. \end{aligned}$$

This inequality holds, if

$$(1+o(1))(\Delta(D) + 2 \cdot e(D[A])/n + 3 \cdot e(D[A, B])/n + 4 \cdot e(D[B])/n) \leq n.$$

Using our observations at the beginning of this proof, the previous inequality is a consequence of the regularity of D and

$$(1+o(1)) \cdot 4 \cdot \Delta(D) \leq n.$$

Thus, if the conditions of the statement of this theorem hold, there is a proper list edge coloring c_3 which maps each $e \in E(D''[A])$ to an element of $L(e)$. Finally, we lift every edge $e \in E(D''[A])$ to $c_3(e)$. As we do not create multiple edges between A and B , the resulting graph is a realization of D . \square

Proof of Theorem 2. This proof is a slight variation on the previous proof. We do not lift edges of $D[A]$ to elements of A , and Shannon's theorem (Theorem 7) will be used to bound the chromatic index of a graph induced by A .

We may assume that D is regular. For the first step, we use Claim 9 to take a proper edge n -coloring c_1 of $D[A, B] \cup D[B]$, which is an equitable n - or $(n-1)$ -coloring if restricted to both $D[A, B]$ and $D[B]$. Lift $c_1^{-1}(i)$ to a_i for all $i = 1, \dots, n$ to get D' from D . Now $D'[A, B]$ is simple and $D'[B]$ is an empty graph on n -vertices. For any $a \in A$ and $b \in B$, we have the following estimates:

$$\begin{aligned} e_{D'}(a, A) &\leq e_D(a, A) + e_D(a, B) + \lceil e(D[A, B])/(n-1) \rceil, \\ e_{D'}(a, B) &\leq \lceil e(D[A, B])/(n-1) \rceil + 2 \cdot \lceil e(D[B])/(n-1) \rceil, \\ e_{D'}(b, A) &= \Delta(D), \quad e_{D'}(b, B) = 0. \end{aligned}$$

To each edge $e \in E(D'[A])$ with end vertices a_i and a_j , we associate a list $L(e)$ of vertices of B , to which we can lift e to without creating multiple edges:

$$L(e) = B \setminus (N_{D'}(a_i) \cup N_{D'}(a_j)),$$

whose size is bounded from below

$$\begin{aligned} |L(e)| &\geq n - e_{D'}(a_i, B) - e_{D'}(a_j, B) \geq \\ &\geq n - 2\lceil e(D[A, B])/(n-1) \rceil - 4 \cdot \lceil e(D[B])/(n-1) \rceil \geq \\ &\geq n - (1+o(1))2\Delta(D). \end{aligned}$$

By Shannon's theorem (Theorem 7),

$$\begin{aligned} \chi'(D'[A]) &\leq \frac{3}{2}\Delta(D'[A]) \leq \\ &\leq \frac{3}{2} \cdot \max_{a \in A} (e_D(a, A) + e_D(a, B) + \lceil e(D[A, B])/(n-1) \rceil) \leq \\ &\leq (1+o(1)) \cdot \frac{3}{2} \cdot (\Delta(D) + e(D[A, B])/n). \end{aligned}$$

Furthermore, by Kahn's theorem (Theorem 8), $\text{ch}'(D'[A]) \leq (1 + o(1))\chi'(D'[A])$. We have $\text{ch}'(D'[A]) \leq |L(e)|$ for each edge e in $E(D'[A])$, if

$$(1 + o(1)) \cdot \frac{3}{2} \cdot (\Delta(D) + e(D[A, B])/n) \leq n - 2\Delta(D).$$

This holds, if

$$(1 + o(1)) \cdot \left(\frac{7}{2} \cdot \Delta(D) + \frac{3}{2} \cdot \frac{e(D[A, B])}{n} \right) \leq n.$$

Thus, if the conditions of the statement of this theorem hold, there is a proper list edge coloring c_2 which maps each $e \in E(D'[A])$ to an element of $L(e)$. Finally, we lift every edge $e \in E(D'[A])$ to $c_2(e)$. As we do not create multiple edges between A and B , the resulting graph is a realization of D . \square

Complexity remarks. The proofs of Theorem 1 and 2 are recipes describing (randomized) algorithms. It is easy to see that apart from constructing list edge colorings and edge colorings of multigraphs, the algorithms are linear time (making the demand graphs regular allows our proofs to be less technical, but is not necessary in an algorithm). Thus, the running time of our algorithms are dominated by the complexity of constructing appropriate (list) edge colorings. An edge coloring using $\lceil \frac{3\Delta}{2} \rceil$ can be constructed in $\mathcal{O}((|V(D)| + |E(D)|) \log \Delta(D))$ using the algorithms of Karloff and Shmoys [22] and Cole, Ost, and Schirra [4], which may be used as a substitute of Shannon's theorem. The proof of Kahn's result [21] is probabilistic, and it can be emulated in polynomial time.

Using greedy edge coloring algorithms, a realization for a demand graph D in $K_{n,n}$ can be computed in deterministic linear time if

$$\Delta(D) \leq \frac{1}{6}(n - 1), \text{ or}$$

$$\Delta(D) \leq \frac{1}{4} \left(n - 2 \cdot \left\lceil \frac{e(D[A, B])}{n - 1} \right\rceil \right).$$

These conditions correspond to Theorem 1 and 2, respectively. Notice that whenever we use Theorem 7 directly to find an edge coloring (that is, when we are not looking for a list edge coloring), even an edge coloring using at most $2\Delta - 1$ colors is sufficient (given the bounds on $\Delta(D)$ in the assumptions of this theorem). Therefore the greedy edge coloring algorithm can be used instead, which runs in $\mathcal{O}(|V(D)| + |E(D)|)$. Again, using the greedy edge coloring algorithm, we can compute an edge coloring for any multigraph G , such that each edge $e \in E(G)$ gets its color from a set $L(e)$ with $|L(e)| \geq d_e + 1$, where d_e denotes the number edges incident to e .

These bounds are not as tight as our theoretical bounds, but are smaller only by a factor of $\frac{3}{2}$ and $\frac{8}{7}$, respectively. Also notable is a recent result of Fischer, Ghaffari, and Kuhn [17], which implies that the (list) edge colorings required by our deterministic algorithms can be computed relatively efficiently even in a distributed system.

3. PROOF OF THE EDGE VERSION (THEOREM 3)

We apply induction on n . It is easy to check the result for $n \leq 3$. Assume from now on that $n \geq 4$ and let D be a demand graph on $2n - 3$ edges (we may assume that by adding edges to the graph, if necessary). Recall that A and B be the color classes of $K_{n,n}$, and let

$$S = \{v \in A \cup B : d_D(v) \geq n - 1\}.$$

Since D has $2n - 3$ edges, it is clear that $|S| \leq 3$ and that for every pair of vertices in S there is at least one edge joining them.

For a vertex $v \in V(D)$, we denote by $d(v)$ its degree and by $\gamma_A(v)$, $\gamma_B(v)$ the number of neighbors of v in class A and B , respectively. Let $d'(v)$, $\gamma'_A(v)$, $\gamma'_B(v)$ denote the value of these quantities after resolution of a vertex in D ; similarly, $d''(v)$, $\gamma''_A(v)$, $\gamma''_B(v)$ denotes the values after the resolution of a second vertex, and so on. We denote the multiplicity of an edge uv by $\mu(uv)$, and we call it *monochromatic* if u and v are in the same color class of D , and *crossing*, otherwise.

Notice that, for a vertex $v \in A$, we need precisely $d(v) - \gamma_B(v)$ vertices in $B \setminus N_B(v)$ (which can be freely chosen in this set) to lift all the multiple edges and monochromatic edges incident to v . After these liftings, which increased the number of edges of the graph by $d(v) - \gamma_B(v)$, all the edges incident to v have their other endpoint in B and are simple. Clearly, we have the same for a vertex in B , exchanging all the occurrences of A and B . We say in this case that v is *resolved*.

For the induction step, we will resolve $t = 1$ or 3 vertices in each color class of D (possibly making some liftings before), remove them from the graph, getting a smaller graph D' , and apply the induction hypothesis on D' . It is clear that D is realizable if D' is. By the inductive hypothesis, D' is realizable if the following conditions hold:

- (1) $\Delta(D') \leq n - t$,
- (2) D' has at most $2(n-t) - 3$ edges, i.e., there were at least $2t$ edges incident to the $2t$ removed vertices after their resolution.

Assume first that there are 3 vertices of degree n in D lying on the same color class (this can only happen if $n \geq 6$, since we must have $3n \leq \sum_{v \in D} d(v) = 4n - 6$).

Let $x, y, z \in A$ be the vertices of degree n . As $e(D) = 2n - 3$, it is clear that we have $\mu(xy) + \mu(xz) + \mu(yz) \geq n + 3$ and that there are at least 6 isolated vertices in B . We choose three from them, say, a, b, c . Without loss of generality, we may assume that $\mu(xy) + \mu(xz) \geq 2/3 \cdot (n + 3) \geq 6$.

We resolve x, y and z in this order. After resolving x , we have $\gamma'_B(y) \geq \mu(xy)$, and after resolving y , we have $\gamma''_B(z) \geq \mu(xz)$. In total, we add $d(x) - \gamma_B(x) + d'(y) - \gamma'_B(y) + d''(z) - \gamma''_B(z)$ edges to D , and we delete at least $d(x) + d'(y) + d''(z)$ edges when we remove x, y, z, a, b, c from D , so we lost at least $\gamma_B(x) + \gamma'_B(y) + \gamma''_B(z) \geq \mu(xy) + \mu(xz) \geq 6$ edges, and we can apply induction since $\Delta(D') \leq n - 3$.

From now on, we may assume that there are at most two vertices of degree n in a class.

Let u be a maximum degree vertex in D . We may assume that $u \in A$. We distinguish some cases based on the value of $\gamma_B(u)$:

Case 1. $\gamma_B(u) \geq 2$, or $\gamma_B(u) = 1$ and $N_A(u) \neq \emptyset$.

We resolve the demands of u first and then a vertex $v \in B$ which was used for a lifting of an edge uu' , where $u' \in A$ has the maximum degree among $N_A(u)$. If $N_A(u) = \emptyset$, we simply choose v arbitrarily among the neighbors of u in B . This choice guarantees that $\Delta(D') \leq n - 1$. The lifting added $d(u) - \gamma_B(u) + d'(v) - \gamma'_A(v)$ edges to D , and we lose $d(u) + d'(v) - 1$ edges when we remove u and v , so we lost $\gamma_B(u) + \gamma'_A(v) - 1 \geq 2$ edges, and we can apply the induction hypothesis on D' .

Case 2. $\gamma_B(u) = 1$ and $N_A(u) = \emptyset$.

Let u' be the neighbor of u in B . If u' has another neighbor distinct from u , we would have $d(u') > d(u)$, a contradiction. So uu' forms a bundle. Also, if there is any crossing edge vv' not belonging to this bundle, resolving u and $v' \in B$ and deleting them, we are done by induction again, since we lose at least two edges, and $\Delta(D') \leq n - 1$.

Assume now that $E(D)$ consists of the bundle uu' and monochromatic edges not incident to u or u' . In this case, take $a \neq u$ in A , $b \neq u'$ in B with smallest degree (by the number of edges, it is at most 3). Take an edge e , say, in A , not incident to a , lift it to b , replace one copy of the edge uu' by the path $ubau'$. Then resolve the multiple edges of a and b , and delete both of them. The remaining graph D' has $\Delta(D') \leq n - 1$ and two less edges than D , so we may apply the induction hypothesis on D' .

Case 3. $\gamma_B(u) = 0$.

We resolve u in a way that we lift an edge uu' , where u' is a maximum degree neighbor of u , to $v \in B$, and then we resolve v . This guarantees that after deleting u and v we get $\Delta(D') \leq n - 1$ and lose $\gamma'_A(v) - 1$ edges. If $\gamma'(v) \geq 3$, we can proceed by induction.

Assume now that $\gamma'_A(v) \leq 2$ for all $v \in B$, which means that $N_A(v) = \emptyset$ or $N_A(v) = \{u'\}$. It is clear the latter cannot happen to every vertex in B , otherwise u' would have degree at least $n+1$, so we have at least one vertex $b \in B$ satisfying $N_A(b) = \emptyset$. We consider three subcases:

Case 3.1. There is a crossing edge in D .

Let xy be such an edge, $x \in A$, $y \in B$ (note that we may have $x = u'$). Lift an edge uu' to y , resolve x and y , and delete them. We get a graph D' with $\Delta(D') \leq n - 1$ and we removed at least $\gamma_B(x) + \gamma_A(y) \geq 2$ edges, so we can apply induction.

Case 3.2. There is no crossing edge in D and there exist e, e' independent edges with $u \in e$ and $u' \in e'$.

Let $a \neq u, u'$ be a vertex in A . Lift both e and e' to b , resolve a , and delete both a and b . We get D' with $\Delta(D') \leq n - 1$ and at most $2n - 5$ edges, so we can apply induction.

Case 3.3. There is no crossing edge in D , and there do not exist e, e' independent edges with $u \in e$ and $u' \in e'$.

Again, let $a \neq u, u'$ be a vertex in A . Lift a copy of the edge $e = uu'$ to b , and take an edge $f = xy$ independent of e . If e is monochromatic in A , lift it to b ; if it is monochromatic in B , lift it to a , resolve a and b , and delete them. The remaining graph D' has $\Delta(D') \leq n - 1$ and at most $2n - 5$ edges, so we may apply the induction hypothesis again.

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