The mixing time of switch Markov chains: a unified approach

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Abstract

Since 1997 a considerable effort has been spent to study the mixing time of **switch** Markov chains on the realizations of graphic degree sequences of simple graphs. Several results were proved on rapidly mixing Markov chains on unconstrained, bipartite, and directed sequences, using different mechanisms. The aim of this paper is to unify these approaches. We will illustrate the strength of the unified method by showing that on any P-stable family of unconstrained/bipartite/directed degree sequences the switch Markov chain is rapidly mixing. This is a common generalization of every known result that shows the rapid mixing nature of the switch Markov chain on a region of degree sequences. Two applications of this general result will be presented. One is an almost uniform sampler for power-law degree sequences with exponent $\gamma > 1 + \sqrt{3}$. The other one shows that the switch Markov chain on the degree sequence of an Erdős-Rényi random graph $G(n,p)$ is asymptotically almost surely rapidly mixing if p is bounded away from 0 and 1 by at least $\frac{5 \log n}{n-1}$.

Keywords: degree sequences, realizations, switch Markov chain, rapidly mixing, MCMC, Sinclair's multi-commodity flow method

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1. Introduction

An important problem in network science is to algorithmically construct typical instances of networks with predefined properties. In particular, special attention has been devoted to sampling simple graphs with a given degree sequence. In this paper only graphs without parallel edges and loops are considered and we restrict our study to degree sequences which have at least one realization (**graphic**). We study the three most common degree sequence types: bipartite degree sequences, directed degree sequences and the usual degree sequences which we call **unconstrained**, or **UC degree sequence** for short.

In 1997 Kannan, Tetali, and Vempala ([25]) proposed the use of the **switch Markov chain** (also known as the swap chain [28]) for uniformly sampling realizations of a degree sequence. For all three degree sequence types, the switch Markov chain can be thought of as the Markov chain of smallest possible modifications. To illustrate this, we give an informal description of the switch Markov chain on UC degree sequences. If G_1, G_2 are two realizations of the same UC degree sequence, it is easy to see that the minimum size of the symmetric difference $E(G_1) \triangle E(G_2)$ is four. We say that G_1 and G_2 differ by a switch if this symmetric difference is exactly four. The states of the switch Markov chain are the realizations of the degree sequence and the probability of going from realization G_1 to G_2 is nonzero if and only if they differ by a switch (and this nonzero quantity is independent of G_1 and G_2). For the precise definition of this chain, and for the definition of the chains for other degree sequence types, we refer the reader to Section 3 (unconstrained and bipartite) and Section 5 (directed).

The following conjecture has been named after Kannan, Tetali, and Vempala, in recognition of their pioneering work.

Conjecture 1.1 (the KTV conjecture)**.** *The switch Markov chain is rapidly mixing for any bipartite, directed, or UC degree sequence.*

To give some context to the Conjecture, we say that a Markov chain is rapidly mixing if the distribution on the state space is close in ℓ_1 norm to the unique stationary distribution after $\mathrm{poly}(\log N)$ steps, where N is the size of the state space. This property means that sampling from the state space with the stationary distribution is a more or less tractable problem, even if the state space has exponential size.

It is not uncommon that uniformly randomly applied, small local modifications of combinatorial objects result in rapid mixing. This is the case for solutions of the 0–1 knapsack problem [29], for the union of perfect and almost perfect matchings of a graph [22], and two-rowed contingency tables [4], for example. In all of these cases, applying the smallest possible modifications of the respective combinatorial objects randomly, result in rapid mixing of the corresponding Markov chain.

Although Conjecture 1.1 is still open, there is a series of results that prove the rapid mixing of the switch Markov chain on various special degree sequence classes.

We summarize these results in a very compact way in Table 1 without presenting the sometimes lengthy definitions of the special classes, which can be found in the references provided. Some rapid mixing results on directed degree sequences work with directed graphs [16], while some work in the bipartite representation with a forbidden perfect matching [8]. Since we use the bipartite representation in the present paper, we will not discuss directed degree sequences until Section 5.

UC degree sequences	bipartite deg. seq.	directed deg. seq.
regular [3]	(half-)regular [28]	regular [16]
	almost half regular [8]	
$\Delta \leq \frac{1}{3}\sqrt{2m}$ [18]	$\Delta \leq \frac{1}{\sqrt{2}}\sqrt{m}$ [11]	$\Delta < \frac{1}{\sqrt{2}}\sqrt{m-4}$ [11]
Power-law density-bound, $\gamma > 2.5$ [18]		
$(\Delta - \delta + 1)^2 \leq$ $\leq 4 \cdot \delta(n - \Delta - 1)$ proof in [1]	$(\Delta - \delta)^2 \leq$ $\leq \delta(\frac{n}{2} - \Delta)$ [10, 11] (or Corollary 18 in [1])	similar to bipartite case [10, 11]
	Bipartite Erdős-Rényi [10, 11] $p,\ 1-p \geq 4\sqrt{\frac{2\log n}{n}}$	similar to bipartite case [10, 11]
strongly stable degree sequence classes [1]		

Table 1: Some classes of degree sequences for which the switch Markov chain is rapidly mixing. Here Δ and δ denote the maximum and minimum degrees, respectively. Half of the sum of the degrees is m, and n is the number of vertices. The notation is similar for bipartite and directed degree sequences. Some technical conditions have been omitted.

Notice, that some, but not all of the results came in pairs for unconstrained and bipartite (directed) degree sequences. The reason for this discrepancy is the following: while both set of results are based on Sinclair's multicommodity flow method, one of them has to deal with special circuits (one of the vertices may be visited at most twice) instead of just cycles in the decomposition of symmetric differences of two realizations of a degree sequence. The main goal of this paper to remedy this discrepancy between the machineries used for the bipartite and unconstrained degree sequences by decomposing into circuits where each vertex is visited at most twice. Along the way we also give new, much more transparent proofs for the main results in [28].

Greenhill and Sfragara suggested exploring the connection between the mixing rate of the switch Markov chain and stable degree sequences [18, Subsection 1.1]. The first notion of stability for unconstrained and bipartite degree sequences was *P-stability*, introduced by Jerrum and Sinclair [23]. An unconstrained degree sequence on n vertices, usually denoted by $\mathbf{d}$, is an element of $[n]^n$. The set of graphs with the degree sequence $\mathbf{d}$ is denoted by $\mathbb{G}(\mathbf{d})$.

Definition 1.2. Let $\mathcal{D}$ be an infinite set of unconstrained degree sequences. We say that $\mathcal{D}$ is ***P*-stable**, if there exists a polynomial $p \in \mathbb{R}[x]$ such that for any $n \in \mathbb{N}$ and any degree sequence $\mathbf{d} \in \mathcal{D}$ on n vertices we have

$$\left| \mathbb{G}(\mathbf{d}) \cup \left(\bigcup_{x,y\in[n],\ x\neq y} \mathbb{G}(\mathbf{d} + \mathbb{1}_x + \mathbb{1}_y) \right) \right| \leq p(n) \cdot |\mathbb{G}(\mathbf{d})|,$$

where $\mathbb{1}_x$ is the x^{th} unit vector.

Without proof we state, that the notion of P-stability does not change even if we require

$$\left| \left\{ G \mid G \in \mathbb{G}(\mathbf{d}'),\ \mathbf{d}' \in \mathbb{N}^n,\ \ell_1(\mathbf{d}, \mathbf{d}') \leq 2 \right\} \right| \leq p(n) \cdot |\mathbb{G}(\mathbf{d})|. \tag{1.1}$$

With a bit of care, Definition 1.2 generalizes to bipartite and directed degree sequences: we require that the realizations of perturbed degree sequences are also bipartite or directed, respectively. In a forthcoming paper we will study P-stability in detail, but here we confine ourselves to the following observation, some statements and their direct consequences.

Amanatidis and Kleer [1] introduced and studied *strong stability* of unconstrained and bipartite degree sequences and proved that strongly stable degree sequences are P-stable. The definition of strong stability is given at the beginning of Section 8.

Informally, a class of UC degree sequences is P-stable if making slight perturbations to the degree sequence from a P-stable class cannot increase the number of realizations too much. Thus P-stability is a property of the degree sequences directly, in the sense that it only cares about the number of their (perturbed) realizations. Strong stability, in contrast, requires the existence of some structure *in each realization* of the perturbed degree sequences. It is not known whether these concepts of stability are equivalent.

Amanatidis and Kleer [1] established rapid mixing of the switch Markov chain on strongly stable unconstrained and bipartite degree sequences. We prove that the switch Markov chain is rapidly mixing for the possibly-larger class of P-stable unconstrained, bipartite, and directed degree sequences, using a unified machinery which covers all previously-known results.

Theorem 1.3 (proved in Section 7)**.** *The switch Markov chain is rapidly mixing on P-stable unconstrained, bipartite, and directed degree sequence classes.*

Until recently, all classes for which Conjecture 1.1 have been proved are strongly stable, or P-stable directed degree sequences. (Strong stability for directed degree sequences is not defined in [1].) However, Erdős et al. [5] describe a class of non-P-stable degree sequences for which the switch chain is rapidly mixing.

One of the technical novelties of this paper is extending the T-operator of [28] from bipartite to unconstrained graphs. The atoms of a decomposition generated by the T-operator are so-called *primitive circuits* (see Definition 2.3), along which we recursively construct the multi-commodity flow required by the Sinclair method. The processing of the primitive circuits via Algorithm 2.1 is in turn an extension of the Sweep algorithm of [28]. These refinements require a fairly extensive and detailed analysis of the structures of the realizations under study. In exchange, the unified framework in which we prove Theorem 1.3 allows us to treat unconstrained, bipartite and directed degree sequences with minimal branching in the proof.

There are two interesting direct consequences of Theorem 1.3 concerning popular unconstrained random graph models. It turns out that asymptotically almost surely, the degree sequence of an Erdős-Rényi random graph $G(n,p)$ is P-stable if p is bounded away from 0 and 1 by $\frac{5 \log n}{n-1}$, see Corollary 8.6. Gao and Wormald [15] show that power-law distribution-bounded degree sequences with parameter $\gamma > 1 + \sqrt{3}$ are P-stable. Consequently, Theorem 1.3 implies that the switch Markov chain is rapidly mixing on these degree sequences: see Section 8.2. This gives the first formal verification of the validity of generating random power-law graphs via the switch Markov Chain [19, 13].

The proof of Theorem 1.3 relies on Sinclair's multicommodity flow method (Section 3), which can be described informally in the case of the switch Markov chain as follows. Suppose that the chain has N states, and let $\mathbb{G}$ be the graph underlying the Markov chain: the vertex set of $\mathbb{G}$ is the state space of the chain (in our applications, the states are realizations of a given degree sequence), and two states are adjacent in $\mathbb{G}$ if the transition probability between them is non-zero in the chain. Sinclair's multicommodity flow method ensures the rapid mixing of the chain if we can design a multicommodity flow on $\mathbb{G}$ which transfers a unit amount of commodities between each pair of vertices (different commodities for different pairs), and no more than $N \cdot \mathrm{poly}(\log(N))$ amount of commodities go through any vertex (no vertex is overloaded). Hence most of the present paper is devoted to the design of a flow and the proof that it does not overload any vertex.

The paper is structured as follows. In Section 2, we give a slightly more detailed description of the flow and we present the necessary graph theoretic tools with which we will use to construct paths that will form the flow. In Section 3 we give the formal definition of Sinclair's multicommodity flow method and its simplified version that is tailored to our needs. In Section 4 we finally describe the mulicommodity flow. In Section 5 we define the switch Markov chain for directed degree sequences. Before turning into the home straight, an auxiliary structure that tracks the defined flow is analyzed in Section 6. In Section 7 we finally prove Theorem 1.3 and we also provide the necessary modifications to deal with bipartite and directed degree sequences. Lastly, we describe the known P-stable regions of degree sequences in Section 8 and present the connections between Theorem 1.3 and the aforementioned popular graph models.

2. Definitions and preliminaries, the structure of the sets of realizations.

Let us recall some well known notions and notations. Let $\mathbf{d} = (d(v_1), \ldots, d(v_n))$ denote a UC degree sequence and let

$$\mathbf{D} = \big((d(u_1), \ldots, d(u_{n_1})), (d(v_1), \ldots, d(v_{n_2}))\big)$$

denote a bipartite degree sequence on the bipartition (U, V). (For convenience we assume that $n_1 \geq n_2$ and that $\mathbf{D}$ can also be considered as an $n_1 + n_2$ long vector.) We will use the notations $\mathbb{G}(\mathbf{d})$ and $\mathbb{G}(\mathbf{D})$ for the sets of all realizations of the corresponding degree sequences.

The **switch operation** exchanges two disjoint edges ac and bd in the realization G with ad and bc if the resulting configuration G' is again a simple graph (we denote the operation by $ac, bd \Rightarrow ad, bc$). In our terminology, a switch is a matrix operation, which will be introduced in Section 6.

For an $ac, bd \Rightarrow ad, bc$ switch operation to be valid, it is necessary but not always sufficient that both $ac, bd \in E(G)$ and $ad, bc \notin E(G)$ hold. For each setting (graphs, bipartite graphs, directed graphs) we will define a set of **non-chords**, which are pairs of vertices which are forbidden from forming edges. See Definition 2.1 below. A pair of vertices which is not a non-chord will be called a **chord**: such pairs are allowed to be edges, so they may be inserted or deleted. We emphasize that whether or not a pair of vertices form a **chord** does not depend on the current realization.

Next, we reformulate the definition of the switch operation to avoid inserting non-chords: an $ac, bd \Rightarrow ad, bc$ switch operation can be applied if $ac, bd \in E(G)$, $ad, bc \notin E(G)$, and ad, bc are both chords. We now define the set of chords and non-chords in the case of unconstrained and bipartite graph models.

Definition 2.1. For simple graphs, the non-chords are exactly the pairs of the form (v, v), as loops are forbidden. Because no further constraints have to be set, we call their degree sequences **unconstrained**. In **bipartite** graphs, (u, v) is a chord if and only if u and v are in different vertex classes.

In the case of directed graphs (Section 5), we further restrict the set of **chords**.

It is a well-known fact that the set of all possible realizations of a **graphic** UC degree sequence is connected under the switch operation. See for example [21] or [20]. It is interesting to know, however, that the first known proof is from 1891 [30]. For bipartite graphs the equivalent results were proved in 1957 in [12] and [31]. The "classical" proofs work through so called "canonical" realizations. However, the paths between different realizations, created in this way, are often very far from shortest possible. Therefore in this paper we will use another way to design these paths.

To that end, let us consider two realizations of the same (bipartite or unconstrained) degree sequence. Those edges, that are present or missing in both realizations, do not need to be changed. The remaining pairs of vertices, each of which is an edge in exactly one of the realizations, form the symmetric difference of the two realizations, usually denoted by ∇. To any alternating circuit decomposition of ∇, we are going to assign a sequence of switches that transform the first realization into the second (this is described right after the proof of Lemma 2.5).

A graph H, with edges colored by either red or blue, will be called a **red-blue graph**. For vertex v let $d_r(v)$ and $d_b(v)$ be the degree of vertex v in red and blue edges, respectively. This red-blue graph is **balanced** if for each $v \in V(H)$ equality $d_r(v) = d_b(v)$ holds.

Let G, G' both be realizations (on the same vertex set) of an unconstrained degree sequence $\mathbf{d}$ or a bipartite degree sequence $\mathbf{D}$. Let the symmetric difference of the edges be

$$\nabla = E(G) \triangle E(G').$$

Color the edges of ∇ according to which graph they come from: the $E(G)$ edges are colored red and the $E(G')$ edges are colored blue. Equipped with this coloring, ∇ is a balanced red-blue graph.

A **circuit** in a graph H is a closed trail (so any edge is traversed at most once). As the graph is simple, a circuit is determined by the sequence of the vertices $v_0, \ldots, v_t$, where $v_0 = v_t$. Note that there can also be other indices $i < j$ such that $v_i = v_j$. A circuit is called a **cycle**, if its simple, i.e., for any $i < j$, $v_i = v_j$ only if $i = 0$ and $j = t$.

A circuit (or, in particular, a cycle) in a balanced red-blue graph is called **alternating**, if the color of its edges alternates. In other words, the color of the edge from v_i to v_{i+1} differs from the color of the edge from v_{i+1} to v_{i+2}, and also edges v_0v_1 and $v_{t-1}v_t$ have different colors. Consequently, alternating circuits have even length. The following observations are easy to see.

Lemma 2.2 (adapted from [6])**.**

(i) *If H is a balanced red-blue graph then the edge set can be decomposed into alternating circuits.*

(ii) *Let $C = v_0, v_1, \ldots, v_{2t} = v_0$ be an alternating circuit in a balanced red-blue graph H, in which for some $i < j < 2t$, $j - i$ is even and $v_i = v_j$. Then the circuit can be decomposed into two shorter alternating circuits $v_i, v_{i+1}, \ldots, v_{j-1}, v_j$ and $v_j, v_{j+1}, \ldots, v_{2t-1}, v_0, v_1, \ldots, v_{i-1}, v_i$.*

It is clearly possible that a vertex occurs twice in an alternating circuit without the possibility to divide it into two, smaller alternating circuits. The smallest example is a "bow-tie" circuit: $v_1, v_2, v_3, v_1, v_4, v_5, v_1$ with an alternating edge coloring. (The very first and very last occurrences of v_1 shows the closing of the alternating circuit.) Recalling our earlier discussion, these two copies of the vertex v_1 form a **non-chord**.

Definition 2.3. An alternating circuit is **primitive**, if it cannot be decomposed further in the way described in Lemma 2.2(ii).

From Lemma 2.1 it follows, that in a primitive alternating circuit, no vertex can appear more than twice: moreover, if a vertex appears twice in a circuit then the distance of two copies of the same vertex must be odd. Therefore, in a bipartite graph, a primitive alternating circuit is an alternating cycle.

The definition of primitive alternating circuits is different from the definition of elementary alternating circuits introduced in [6]. Without providing the definition here, we mention that, for example, the red-blue graph obtained from a red C_5 and its blue complement is a primitive alternating circuit, see Figure 1. Still, it can be decomposed into an alternating C_4 (x_1, x_5, x_7, x_6) and an alternating bow-tie, so it is not elementary.

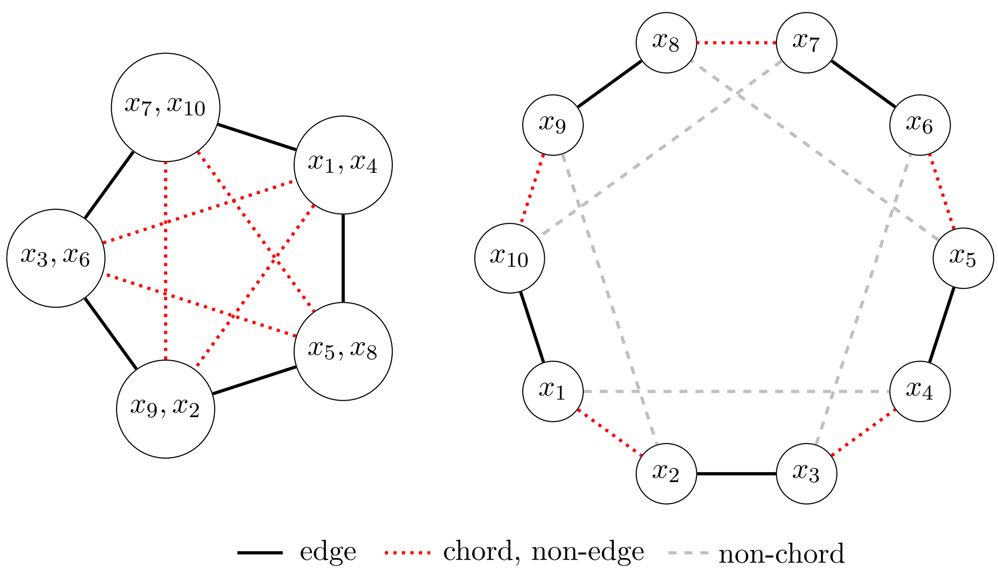

Figure 1: The cycle of length 5 has an alternating circuit traversing all of its edges and non-edges

Lemma 2.4. *Let C be an alternating circuit of length 6 in ∇. If loops are non-chords, then there is at most one vertex which is visited more than once by C.*

Proof. If v is a vertex that is visited at least twice by C, it has at least four other neighbors that are pairwise distinct from each other and v. Since v is counted twice in the length of C, every vertex of C is accounted for, and the claim holds (and C is a bow-tie). $\square$

We will use Sinclair's multicommodity flow method (Theorem 3.2) to bound the mixing time of the switch Markov chain. The multicommodity flow is given by a set of switch sequences between any two realizations of the degree sequence. The main idea behind the definition of the flow can be described roughly as follows:

For each pair of realizations (X, Y) and every possible complete matching of the X-edges with Y-edges in $\nabla = X \triangle Y$ at every vertex, assign a path (a switch sequence) as follows. The matchings decompose ∇ into alternating circuits (Lemma 4.1). Refine each of the alternating circuits into primitive alternating circuits in a canonical way (Section 4). By concatenating the switch sequences given by Algorithm 2.1 for the primitive alternating circuits, a switch sequence from X to Y is obtained.

Here we reached a very important point: Algorithm 2.1 does not require an order for processing the primitive circuits; in principle it can be done arbitrarily. One of the novelties of this paper leading to the unified proof is constructing a very delicate order of processing the primitive circuits, which ultimately enables the unificiation of the proofs. We will return to this point in Section 4.

The Sweep procedure in Algorithm 2.1 will be used to construct the switch sequence between two realizations whose symmetric difference is a primitive alternating circuit $x_1 x_2 \cdots x_{2\ell}$. We assume that $x_1 x_2 \notin E(G)$. Most steps of the algorithm involve a single switch (line 29) which removes an edge $x_1 x_{2t+2}$, fixes two edges $x_{2t} x_{2t+1}$, $x_{2t+1} x_{2t+2}$ of the symmetric difference, and inserts a chord $x_1 x_{2t}$. If $x_1 x_{2t}$ is not a chord then a different function, Double step (described in Algorithm 2.2), is called twice in succession, to perform two switches. Double step is designed to avoid the use of $x_1 x_{2t}$, and is also applied in a couple of other tricky situations (captured by the set $\mathcal{M}$). When $x_1 = x_{2t}$ and $x_{2t-1} = x_{2t+2}$ or $x_{2t-2} = x_{2t+1}$, however, calling Double step is not feasible, therefore a special switch avoiding $x_1 x_{2t}$ is performed (lines 19 and 21). Addition and subtraction of edges naturally means that we add the edge to, or remove the edge from, the edge set of the first operand.

Algorithm 2.1 Sweeping a primitive alternating circuit $(x_1, x_2, \ldots, x_{2\ell})$. The algorithm assumes that $x_1x_2 \notin E(G)$.

procedure $\textsc{Sweep}(G, [x_1, x_2, \ldots, x_{2\ell}]) \to [Z_1, Z_2, \ldots, Z_{\ell-2}, \langle Z_{\ell-1}\rangle]$
 $Z_0 \leftarrow G$
 $q \leftarrow 1$
 $\mathit{end} \leftarrow 2$
 if $\exists r \in \mathbb{N}\ x_1 = x_{2r}$ **then**
  $\mathcal{L} \leftarrow \{2i \in 2\mathbb{N} :\ 4 \le 2i \le 2\ell,\ x_1x_{2i} \in E(G) \text{ and } x_{2i} \neq x_{2r+1}\}$
  $\mathcal{M} \leftarrow \{2t \in 2\mathbb{N} \ :\ (2t > 2r \wedge x_{2t} = x_{2r+1}) \vee (2t < 2r \wedge x_{2t} = x_{2r-1})\}$
  $\mathcal{M} \leftarrow \mathcal{M} \cup \{2r\}$
 else
  $\mathcal{L} \leftarrow \{2i \in 2\mathbb{N} :\ 4 \le 2i \le 2\ell,\ x_1x_{2i} \in E(G)\}$
  $\mathcal{M} \leftarrow \emptyset$
 end if

 while $\mathit{end} < 2\ell$ **do**
  $\mathit{start} \leftarrow \min\left\{2i \in \mathcal{L} \ :\ 2i > \mathit{end}\right\}$
  $2t \leftarrow \mathit{start} - 2$

  while $2t \ge \mathit{end}$ **do**
   if $2t \in \mathcal{M}$ **then**
    if $x_1 = x_{2t}$ and $x_{2t+2} = x_{2t-1}$ **then**
     $Z_q \leftarrow Z_{q-1} - \{x_{2t}x_{2t+1}, x_{2t-2}x_{2t-1}\} + \{x_{2t+1}x_{2t+2}, x_1x_{2t-2}\}$
    else if $x_1 = x_{2t}$ and $x_{2t+1} = x_{2t-2}$ **then**
     $Z_q \leftarrow Z_{q-1} - \{x_1x_{2t+2}, x_{2t-2}x_{2t-1}\} + \{x_{2t+1}x_{2t+2}, x_{2t-1}x_{2t}\}$
    else
     $Z_q \leftarrow \textsc{Double step}(Z_{q-1}, x_1, [x_{2t-2}, \ldots, x_{2t+2}])$
     $q \leftarrow q + 1$
     $Z_q \leftarrow \textsc{Double step}(Z_{q-1}, x_1, [x_{2t-2}, \ldots, x_{2t+2}])$
    end if
    $2t \leftarrow 2t - 2$
   else if $2t \notin \mathcal{M}$ **then**
    $Z_q \leftarrow Z_{q-1} - \left\{x_1x_{2t+2}, x_{2t}x_{2t+1}\right\} + \left\{x_1x_{2t}, x_{2t+1}x_{2t+2}\right\}$
   end if
   $q \leftarrow q + 1$
   $2t \leftarrow 2t - 2$
  end while

  $\mathit{end} \leftarrow \mathit{start}$
 end while

end procedure

Lemma 2.5. *Suppose that $E(\nabla)$ is a primitive alternating circuit C of length 2ℓ, and $(x_1, x_2, \ldots, x_{2\ell})$ is a list of its vertices in the order they are visited and $x_1x_2 \notin E(G)$. Then Algorithm 2.1 provides a valid switch sequence of length $\ell-2$ or $\ell-1$ between G and G' in the case of unconstrained and bipartite graph models.*

Proof. The processing done by Algorithm 2.1 is governed by two nested loops. The outer loop iterates the variable *start* through

$$\mathcal{L} = \Big\{2i \in 2\mathbb{N} :\ 4 \le 2i \le 2\ell,\ x_1x_{2i} \in E(G)\Big\} \cap \Big\{2i \in 2\mathbb{N} \mid x_{2i} \neq x_{2r+1} \text{ if } x_1 = x_{2r}\Big\}$$

in increasing order. The set $\mathcal{L}$ is not empty since $x_1x_{2\ell} \in E(G)$ and, if $x_1 = x_{2r}$ for some r then $x_{2\ell} \neq x_{2r+1}$ (each chord is traversed at most once by ∇). In the first iteration, $end = 2$, and in the successive iterations end takes the value taken by $start$ in the previous iteration.

The inner loop performs a series of switches that changes the status of the edges and non-edges induced by consecutive vertices in the interval of vertices between $x_{start}, \ldots, x_{end}$. As a side effect, it also changes chords induced by x_1 and one of the vertices from $\{x_4, x_6, \ldots, x_{2\ell-4}, x_{2\ell-2}\}$, and perhaps some other chords induced by vertices of C, when $2t \in \mathcal{M}$. We have to check that each time a new graph is obtained from Z_{q-1} (lines 19, 21, 23, 25, and 29 in Algorithm 2.1), the changes correspond to valid switches.

Observe the following:

(1) If $x_i = x_j$, then either $i = j$ or $i \not\equiv j \pmod 2$, because of Lemma 2.2(ii). In particular, $\{x_1, x_3, \ldots, x_{2\ell-1}\}$ and $\{x_2, x_4, \ldots, x_{2\ell}\}$ are both sets of size ℓ, and the set of all pairs of equal vertices describes a (partial) matching between the two sets.

(2) If x_1x_{2i} is a chord for some $1 < i < \ell$, and it also happens to be an element of $E(C)$ (it may or may not be an edge in $E(G)$), then $\exists r \in \mathbb{N}$ such that $x_1 = x_{2r}$ and $x_{2i} \in \{x_{2r-1}, x_{2r+1}\}$.

Let us first check the case when G is bipartite. As we remarked earlier, C must be a cycle. If $x_i = x_j$ and $i \neq j$, then observation (1) implies that $i \neq j \pmod 2$, but because C is alternating, x_i and x_j have to be in different vertex classes. Therefore lines 19, 21, 23, 25 in Algorithm 2.1 are never reached: line 29 alone produces the switch sequence. We will use induction to prove that whenever a new iteration of the inner loop (line 16) starts, we have

$$\begin{aligned} Z_{q-1} = G \triangle \{x_ix_{i+1}\ :\ i \in [1, end) \cup [2t+2, start)\} \triangle \\ \triangle \{x_1x_{end}, x_1x_{start}\} \triangle \{x_1x_{2t+2}\}. \end{aligned} \tag{2.1}$$

For $q = 1$, the equation holds. Now assume that $q \ge 2$ and (2.1) holds. The set $\{x_1, x_{2t}, x_{2t+1}, x_{2t+2}\}$ is guaranteed to be a set of four vertices, and Equation (2.1) guarantees that the chords are in Z_{q-1} are in the correct state, i.e., the switch on line 29 neither creates a multi-edge nor deletes an edge which is already not present. Hence $Z_q = Z_{q-1}\triangle\{x_1x_{2t},\ x_{2t}x_{2t+1},\ x_{2t+1}x_{2t+2},\ x_1x_{2t+2}\}$ which implies that (2.1) holds for Z_q (as $2t+2$ becomes $2t$). When $2t = \mathit{end}$, the graph assigned to Z_q on line 29 is $G\triangle\{x_ix_{i+1}\ :\ i \in [1, \mathit{start})\}\triangle\{x_1x_{\mathit{start}}\}$. In particular, $Z_{\ell-1} = G\triangle C = G'$, which is what we wanted.

Next, let us verify the algorithm for unconstrained degree sequences. There are two types of problems that can occur in the unconstrained case that need to be discussed. It is possible that x_1x_{2t} is not a chord (because $x_1 = x_{2t}$), thus it cannot participate in a switch. This problem in itself can be fixed by two calls to Double step (called on lines 23 and 25). If visiting $x_1, x_{2t-2}, x_{2t-1}, x_{2t}, x_{2t+1}, x_{2t+2}, x_1$ in this order traces out an alternating circuit then Lemma 2.4 implies that the conditions of at least one of the two main branches of Double step are satisfied, and hence Double step performs a valid switch. (This section of C is alternating unless one of the edges $x_{2t}x_{2t-1}$ or $x_{2t}x_{2t+1}$ equals one of the chords x_1x_{2i} which has been temporarily altered during the sweep: this situation is discussed below.)

Algorithm 2.2 The Double step avoids inserting or deleting x_1x_{2t}

```
function DOUBLE STEP(Z, x_1, [x_{2t-2}, x_{2t-1}, x_{2t}, x_{2t+1}, x_{2t+2}])
    if x_{2t-2}x_{2t+1} is a chord then
        if x_{2t-2}x_{2t+1} ∈ E(Z) then
            return Z + {x_1x_{2t-2}, x_{2t+1}x_{2t+2}} − {x_{2t-2}x_{2t+1}, x_1x_{2t+2}}
        else if x_{2t-2}x_{2t+1} ∉ E(Z) then
            return Z + {x_{2t-2}x_{2t+1}, x_{2t-1}x_{2t}} − {x_{2t-2}x_{2t-1}, x_{2t}x_{2t+1}}
        end if
    else if x_{2t-1}x_{2t+2} is a chord then
        if x_{2t-1}x_{2t+2} ∈ E(Z) then
            return Z + {x_{2t+1}x_{2t+2}, x_{2t-1}x_{2t}} − {x_{2t-1}x_{2t+2}, x_{2t}x_{2t+1}}
        else if x_{2t-1}x_{2t+2} ∉ E(Z) then
            return Z + {x_1x_{2t-2}, x_{2t-1}x_{2t+2}} − {x_{2t-1}x_{2t+2}, x_1x_{2t-2}}
        end if
    end if
end function
```

The other problem arises when x_1x_{2t} (where $2 < 2t < 2\ell$) is traversed by circuit C, which is the case above in observation (2). See Figure 1, for example. In this situation, we choose not to perform the standard switch from line 29 (although in fact this switch would be valid if x_1x_{2t} is a non-edge when the sweep reaches it). Instead, we want to perform Double step to avoid using x_1x_{2t} in the switch. We are able to perform Double step unless $x_{2t+1}x_{2t-2}$ or $x_{2t-1}x_{2t+2}$ is a non-chord: in these cases, we perform a special switch, either on line 19 or on line 21.

Because any chord is traversed at most once by the circuit C, we have $2, 2\ell \notin \mathcal{M}$. For the same reason, $\mathcal{M} \neq \{2r+2, 2r, 2r-2\}$. Furthermore, $\mathcal{L} \cap \mathcal{M} = \emptyset$: if $x_{2i} = x_{2r-1}$, then $x_1 x_{2r-1} = x_{2r-1}x_{2r} \notin E(G)$.

Again, we claim that whenever a new iteration of the inner loop (line 16) starts, Equation (2.1) holds. We prove this by induction on q. The inductive step is easy to check if Z_q is produced by line 19, line 21 or line 29. Let

$$F = \begin{cases} \{x_i x_{i+1} : i \in [1, \mathit{end}) \cup [2t+1, \mathit{start})\}, & \text{if } x_{2t-2} \neq x_{2t+1} \wedge x_{2t-2}x_{2t+1} \in E(G), \\ \{x_i x_{i+1} : i \in [1, \mathit{end}) \cup [2t-2, \mathit{start})\}, & \text{if } x_{2t-2} \neq x_{2t+1} \wedge x_{2t-2}x_{2t+1} \notin E(G), \\ \{x_i x_{i+1} : i \in [1, \mathit{end}) \cup [2t-1, \mathit{start})\}, & \text{if } x_{2t-2} = x_{2t+1} \wedge x_{2t+2}x_{2t-1} \in E(G), \\ \{x_i x_{i+1} : i \in [1, \mathit{end}) \cup [2t+2, \mathit{start})\}, & \text{if } x_{2t-2} = x_{2t+1} \wedge x_{2t+2}x_{2t-1} \notin E(G); \end{cases}$$

$$H = \begin{cases} \{x_1 x_{2t-2}, x_{2t-2}x_{2t+1}\}, & \text{if } x_{2t-2} \neq x_{2t+1} \wedge x_{2t-2}x_{2t+1} \in E(G), \\ \{x_1 x_{2t+2}, x_{2t-2}x_{2t+1}, x_{2t+1}x_{2t+2}\}, & \text{if } x_{2t-2} \neq x_{2t+1} \wedge x_{2t-2}x_{2t+1} \notin E(G), \\ \{x_1 x_{2t+2}, x_{2t+2}x_{2t-1}\}, & \text{if } x_{2t-2} = x_{2t+1} \wedge x_{2t+2}x_{2t-1} \in E(G), \\ \{x_1 x_{2t-2}, x_{2t+2}x_{2t-1}, x_{2t-2}x_{2t-1}\}, & \text{if } x_{2t-2} = x_{2t+1} \wedge x_{2t+2}x_{2t-1} \notin E(G). \end{cases} \tag{2.2}$$

The following equation holds when the first call to DOUBLE STEP on line 23 is called in an iteration of the inner loop:

$$Z_q = G \triangle F \triangle \{x_1 x_{\mathit{end}}, x_1 x_{\mathit{start}}\} \triangle H. \tag{2.3}$$

(As a representative example, suppose that $x_{2t-2}x_{2t+1}$ is a chord in $E(G)$. Then $Z_q = Z_{q-1} \triangle \{x_1 x_{2t-2}, x_{2t+1}x_{2t+2}, x_{2t-2}x_{2t+1}, x_1 x_{2t+2}\}$ and substituting for Z_{q-1} using (2.1) shows that (2.3) holds.) The second call to DOUBLE STEP on line 25 sorts this mess out. When line 25 is executed:

$$\begin{aligned} Z_q = G \triangle \{x_i x_{i+1} \ : \ i \in [1, \mathit{end}) \cup [2t-2, \mathit{start})\} \triangle \\ \triangle \{x_1 x_{\mathit{end}}, x_1 x_{\mathit{start}}\} \triangle \{x_1 x_{2t-2}\}, \end{aligned} \tag{2.4}$$

which is what we want (by the end of the iteration, $2t$ is decreased by 4 and q increases by 1). If line 19 or 21 is reached during SWEEP, then $Z_{\ell-2} = G'$, otherwise $Z_{\ell-1} = G'$. □

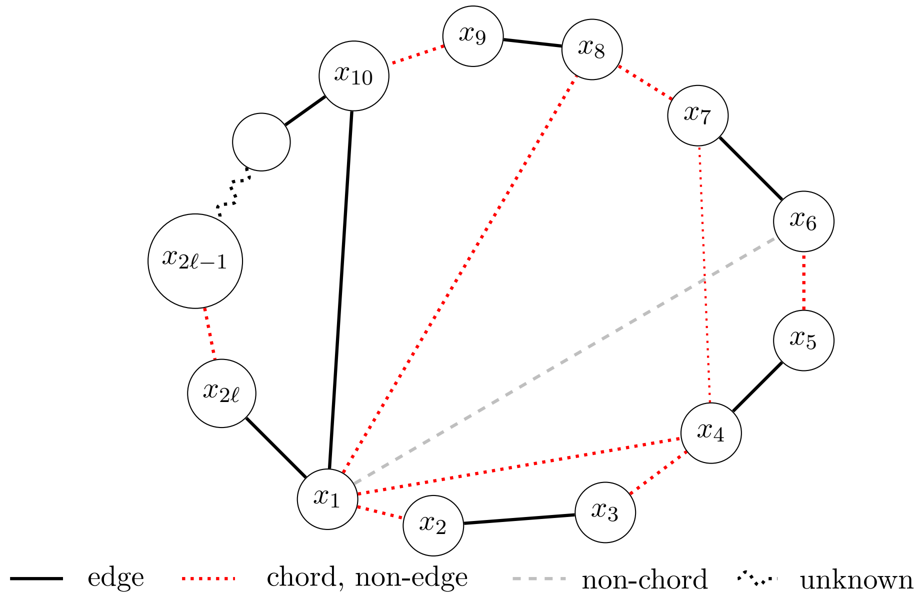

Figure 2: Sweeping a cycle

We will demonstrate the algorithm on Figure 2. In the first iteration of the outer loop, *start* takes 10 as its value. We call x_1x_{10} the **start-chord** and x_1x_2 the **end-chord**. The algorithm **sweeps** the alternating chords along the circuit between x_2 and x_{10}, and vertex x_1 will be the **cornerstone** of this procedure.

The inner loop works from the start-chord x_1x_{10} (edge) towards the end-chord x_1x_2 (non-edge). The first value taken by $2t$ is 8. Since x_1x_8 is a chord, Z_1 is obtained by switching along x_1, x_8, x_9, x_{10}. In the next step, $2t = 6$. However, x_1x_6 is a non-chord, therefore SWEEP calls DOUBLE STEP instead of SWAP. Because x_4x_7 is not an edge, Z_2 is obtained by switching along x_4, x_5, x_6, x_7, and subsequently Z_3 is obtained by switching along x_1, x_4, x_7, x_8. The last iteration of the inner loop switches along x_1, x_2, x_3, x_4 and produces Z_4. Notice, that all of the chords on the circuit from x_2 to x_{10} changed their status and x_1x_{10} is no longer an edge (that is, in Z_4), but the rest of the chords have the same status in Z_4 as they had in G.

For the second iteration of the outer loop, *end* $= 10$ and *start* is assigned a new value too. Eventually, *start* $= 2\ell$, which marks the last iteration of the outer loop, at the end of the algorithm produces $Z_{\ell-1} = G \triangle \nabla = G'$.

The demonstration shows that some chords that are not necessarily in ∇ change from being an edge to a non-edge and vice versa during this procedure. However, there are strict patterns that these irregularities must abide, as shown by the next lemma.

Lemma 2.6. *Suppose Z_q is an intermediate realization produced by Algorithm 2.1 on the switch-sequence between G and G', when $\nabla = E(G) \triangle E(G') = (x_1, x_2, \ldots, x_{2\ell})$*

is a primitive alternating circuit. Let

$$R = ((Z_q \triangle G) \setminus \nabla) \cup Q, \tag{2.5}$$

where $Q = \emptyset$, except if Z_q is the return value of a call to Double step *on line 23, in which case*

$$Q = \begin{cases} \{x_{2t+1}x_{2t+2}\}, & \textit{if } x_{2t-2} \neq x_{2t+1} \textit{ and } x_{2t-2}x_{2t+1} \notin E(G), \\ \{x_{2t-2}x_{2t-1}\}, & \textit{if } x_{2t-2} = x_{2t+1} \textit{ and } x_{2t-1}x_{2t+2} \in E(G), \\ \emptyset, & \textit{otherwise.} \end{cases}$$

The following statements hold at the moment when Z_q is assigned a value in Sweep*.*

(a) R is a set of chords induced by vertices of ∇,

(b) $R = \emptyset$, if $Z_q \in \{G, G'\}$,

(c) $R \subseteq \{x_1x_{start}, x_1x_{end}, x_1x_{2t}\}$, if Z_q is produced on line 29,

(d) $R \subseteq \{x_1x_{start}, x_1x_{end}, x_1x_{2t-2}\}$, if Z_q is produced on line 19, line 21, or line 25 (the second call to Double step*),*

(e) $R \subseteq \{x_1x_{start}, x_1x_{end}\} \cup H$, where H is defined in Equation (2.2)*, if Z_q is the return value of a call to* Double step *on line 23.*

(f) $Z_q \triangle G \triangle R$ is a set of at most 2 edge-disjoint subtrails of ∇, starting and ending at endpoints of chords in R.

(g) The edges in R cover at most 5 vertices besides x_1. In the bipartite case, R covers at most 3 vertices other than x_1.

Proof. This follows almost completely from the proof of Lemma 2.5. In addition, we need to check that $Q \cap E(Z_q \Delta G) = \emptyset$ to verify (f): this follows as the single chord in Q (if non-empty) is not involved in the Double step which created Z_q. (Without the inclusion of the set Q in R, in this case, $Z_q \triangle G \triangle R$ might consist of 3 edge-disjoint subtrails.) □

Let G and G' be two realizations. Assume that we can decompose the symmetric difference $\nabla = G \triangle G'$ into k primitive circuits (cycles). Then Sweep can process all primitive alternating circuits one by one, therefore it can transform G into G' with at most $\frac{|\nabla|}{2} - k$ switch operations. The process only changes the statuses of chords induced by vertices of the circuits.

3. Sinclair's multicommodity flow method

For UC degree sequences we define our Markov chain $(\mathbb{G}_{\mathbf{d}}, P_{\mathbf{d}})$ as follows: in the Markov graph $\mathbb{G}_{\mathbf{d}}(\mathbb{G}(\mathbf{d}), \mathbb{E}_{\mathbf{d}})$ the pair (G, G') is an edge if these two realizations differ in exactly one switch. To make a move, choose an unordered pair F of unordered pairs of distinct vertices, uniformly at random from G, say $F = \{(x,y),(z,w)\}$ and choose a perfect matching F' from the other two perfect matchings on the same four vertices. If $F \subseteq E(G)$ and $F' \cap E(G) = \emptyset$, then perform the switch (so $E(G') = (E(G) \cup F') \setminus F$). Assuming that $P(G,G') \neq 0$ and $G \neq G'$, we have

$$\mathrm{Prob}(G \to G') = P(G,G') := \frac{1}{2\binom{n}{2}\binom{n-2}{2}}. \tag{3.1}$$

Equation (3.1) immediately gives that the Markov chain is symmetric. Notice, that if (F, F') corresponds to a feasible switch, then (F', F) does not, therefore $P(G,G) \geq \frac{1}{2}$ for any realization G. A Markov chain possessing this property of staying in the current state with probability at least $\frac{1}{2}$ is called **lazy**. Laziness implies that the eigenvalues of the transition matrix of the Markov chain are non-negative, and it also implies that the chain is aperiodic.

For bipartite degree sequences we define our Markov chain $(\mathbb{G}_{\mathbf{D}}, P_{\mathbf{D}})$ as follows: in the Markov graph $\mathbb{G}_{\mathbf{D}}(\mathbf{V}_{\mathbf{D}}, \mathbb{E}_{\mathbf{D}})$ the pair (G, G') is an edge, if these two realizations differ in exactly one switch. The transition matrix P is defined as follows: we choose uniformly an unordered pair of distinct vertices from U, and an unordered pair of distinct vertices from V, then uniformly randomly choose one of the two matchings between these two pairs. If it preserves the degree sequence, we remove the chosen matching, and add the other. The switch moving from G to G' is unique, therefore the probability of this transformation (the *jumping probability* from G to $G' \neq G$) is:

$$\mathrm{Prob}(G \to G') = P(G,G') := \frac{1}{2\binom{n}{2}\binom{m}{2}}. \tag{3.2}$$

The transition probabilities are time- and edge-independent, and symmetric. Also, the entries in the main diagonal are at least $\frac{1}{2}$, so the chain is lazy.

To start with we recall some definitions and notations from the literature. Since the stationary distribution of the switch Markov chain is the uniform distribution, we will not state the results we use in full generality. Let P^t denote the t^{th} power of the transition probability matrix and define

$$\Delta_X(t) := \frac{1}{2} \sum_{Y \in V(\mathbb{G})} \left| P^t(X,Y) - 1/N \right|,$$

where X is an element of the state space of the Markov chain and N is the size of the state space. We define the *mixing time* as

$$\tau_X(\varepsilon) := \min_t \left\{ \Delta_X(t') \leq \varepsilon \text{ for all } t' \geq t \right\}.$$

Our Markov chain is said to be *rapidly mixing* if and only if

$$\tau_X(\varepsilon) \le \mathcal{O}\Big(\operatorname{poly}\big(\log(N/\varepsilon)\big)\Big)$$

for any X in the state space. In this case the switch Markov chain method provides is a *fully polynomial almost uniform sampler* (FPAUS) of the realizations of the given degree sequences. Note, that Jerrum and Sinclair have shown that realizations of P-stable degree sequences have a fully polynomial almost uniform sampler [23].

Consider the different eigenvalues of P in non-increasing order: since the Markov chain is lazy, we have

$$1 = \lambda_1 > \lambda_2 \ge \cdots \ge \lambda_N \ge 0.$$

The *relaxation time* τ_{rel} is defined as

$$\tau_{\text{rel}} = \frac{1}{1-\lambda^*}$$

where λ^* is the *second largest eigenvalue modulus*. So $\tau_{\text{rel}} = 1/(1-\lambda_2)$ for lazy chains. The following result was proved implicitly by Diaconis and Strook in 1991, and explicitly stated by Sinclair [32, Proposition 1]:

Theorem 3.1 (Sinclair)**.** $\tau_X(\varepsilon) \le \tau_{rel} \cdot \log(N/\varepsilon)$. □

So one way to prove that our Markov chain is rapidly mixing is to find a $\operatorname{poly}(\log N)$ upper bound on τ_{rel}. We need rapid convergence of the process to the stationary distribution otherwise the method cannot be used in practice.

There are several different methods to prove fast convergence, here we use — similarly to [25] — Sinclair's *multicommodity flow method* [32, Theorem 5'].

Theorem 3.2 (Sinclair)**.** *Let $\mathbb{H}$ be a graph whose vertices represent the possible states of a time reversible finite state Markov chain $\mathcal{M}$, and where $(U,V) \in E(\mathbb{H})$ if and only if the transition probabilities of $\mathcal{M}$ satisfy $P(U,V)P(V,U) \neq 0$. For all $X \neq Y \in V(\mathbb{H})$ let $\Gamma_{X,Y}$ be a set of paths in $\mathbb{H}$ connecting X and Y and let $\pi_{X,Y}$ be a probability distribution on $\Gamma_{X,Y}$. Furthermore let*

$$\Gamma := \bigcup_{X \neq Y \in V(\mathbb{H})} \Gamma_{X,Y}$$

where the elements of Γ are called **canonical paths***. We also assume that there is a stationary distribution π on the vertices $V(\mathbb{H})$. We define the capacity of an edge $e = (W,Z)$ as*

$$Q(e) := \pi(W)P(W,Z)$$

and we denote the length of a path γ by $|\gamma|$. Finally let

$$\kappa_\Gamma := \max_{e \in E(\mathbb{H})} \frac{1}{Q(e)} \sum_{\substack{X,Y \in V(\mathbb{H}) \\ \gamma \in \Gamma_{X,Y} \,:\, e \in \gamma}} \pi(X)\pi(Y)\pi_{X,Y}(\gamma)|\gamma|. \tag{3.3}$$

Then

$$\tau_{\mathrm{rel}}(\mathcal{M}) \le \kappa_\Gamma \tag{3.4}$$

holds. □

We are going to apply Theorem 3.2 for $(\mathbb{G}, P)$, which is either the unconstrained $(\mathbb{G}, P) = (\mathbb{G}(\mathbf{d}), P_{\mathbf{d}})$ or the bipartite $(\mathbb{G}, P) = (\mathbb{G}(\mathbf{D}), P_{\mathbf{D}})$ switch Markov chain. Using the notation $N := |V(\mathbb{G})|$, the (uniform) stationary distribution has the value $\pi(X) = 1/N$ for each vertex $X \in V(\mathbb{G})$. Furthermore each transition probability has the property $P(X,Y) \ge 1/n^4$ (see (3.1) and (3.2)). So if we can design a multicommodity flow such that each path is shorter than an appropriate poly(n) function, then simplifying inequality (3.3) we can turn inequality (3.4) to the form:

$$\tau_{\mathrm{rel}} \le \frac{\mathrm{poly}(n)}{N} \left(\max_{e \in E(\mathbb{H})} \sum_{\substack{X,Y \in V(\mathbb{H}) \\ \gamma \in \Gamma_{X,Y} \,:\, e \in \gamma}} \pi_{\mathrm{X,Y}}(\gamma) \right). \tag{3.5}$$

If $Z \in e$, then

$$\sum_{\substack{X,Y \in V(\mathbb{H}) \\ \gamma \in \Gamma_{X,Y} \,:\, e \in \gamma}} \pi_{\mathrm{X,Y}}(\gamma) \le \sum_{\substack{X,Y \in V(\mathbb{H}) \\ \gamma \in \Gamma_{X,Y} \,:\, Z \in \gamma}} \pi_{\mathrm{X,Y}}(\gamma), \tag{3.6}$$

so we have

$$\tau_{\mathrm{rel}} \le \frac{\mathrm{poly}(n)}{N} \left(\max_{Z \in V(\mathbb{H})} \sum_{\substack{X,Y \in V(\mathbb{H}) \\ \gamma \in \Gamma_{X,Y} \,:\, Z \in \gamma}} \pi_{\mathrm{X,Y}}(\gamma) \right). \tag{3.7}$$

We make one more assumption. Namely, that for each pair of realizations $X, Y \in V(\mathbb{G})$ there is a non-empty finite set $S_{X,Y}$ (which draws its elements from a pool of symbols) and for each $s \in S_{X,Y}$ there is a path $\Upsilon(X,Y,s)$ from X to Y such that

$$\Gamma_{X,Y} = \{\Upsilon(X,Y,s) : s \in S_{X,Y}\}. \tag{3.8}$$

It can happen that $\Upsilon(X,Y,s) = \Upsilon(X,Y,s')$ for $s \ne s'$, so we consider $\Gamma_{X,Y}$ as a "multiset" and so we should take

$$\pi_{X,Y}(\gamma) = \frac{\left|\left\{s \in S_{X,Y} : \gamma = \Upsilon(X,Y,s)\right\}\right|}{|S_{X,Y}|}$$

for $\gamma \in \Gamma_{X,Y}$.

Putting together the observations and simplifications above we obtain:

3.1. The simplified Sinclair's method

There exists a polynomial $\mathrm{poly}_{\mathcal{D}} \in \mathbb{R}[x]$ (which only depends on the degree sequence class $\mathcal{D}$) such that for each $X \neq Y \in V(\mathbb{G})$ find a non-empty finite set $S_{X,Y}$ and for each $s \in S_{X,Y}$ find a path $\Upsilon(X,Y,s)$ from X to Y such that

- each path is shorter than an appropriate $\mathrm{poly}_{\mathcal{D}}(n)$ function,
- for each $Z \in V(\mathbb{G})$

$$\sum_{X,Y \in V(\mathbb{G})} \frac{\left|\left\{s \in S_{X,Y} : Z \in \Upsilon(X,Y,s)\right\}\right|}{|S_{X,Y}|} \leq \mathrm{poly}_{\mathcal{D}}(n) \cdot N. \tag{3.9}$$

Then our Markov chain $(\mathbb{G}, P)$ is rapidly mixing.

4. Multicommodity flow

Let X and Y be two realizations of the same (unconstrained or bipartite) degree sequence; they belong to $\mathbb{G}$. A high level description of the definition of multicommodity flow from X to Y can go like this:

(Step 1) We decompose the symmetric difference $\nabla = E(X) \triangle E(Y)$ into alternating circuits: $W_1, W_2 \ldots, W_p$.

(Step 2) We decompose every alternating circuit W_k into smaller, "simple" alternating circuits $C_1^k, C_2^k \ldots, C_{\ell_k}^k$.

(Step 3) We will construct the canonical path from X to Y along these "simple" alternating circuits, step by step, using Algorithm 2.1 iteratively.

Typically, successful application of Sinclair's method requires decomposing ∇ into alternating circuits (Step 1) in very many ways, and each decomposition requires one canonical path. These different decompositions will be parameterized by the set $S_{X,Y}$ (see (3.8)). This parametrization (described in details in Lemma 4.1) and its application to (Step 1) was introduced in [25]. Now we arrived at the most sensitive part of the construction: (Step 2). Here we need the following ability:

Let Z denote an arbitrary vertex along a canonical path. To apply Sinclair's method we will need that the elements of $S_{X,Y}$ can be reconstructed from elements of $S_{\nabla \cap E(Z), \nabla \setminus E(Z)}$ (using another small parameter set). In case of UC degree sequences (in the current setting, see for example [3] or [18]) these simpler alternating circuits have the following property: in each "simple" circuit C there is one predefined vertex (actually, the smallest vertex in a predefined vertex order), which occurs at most twice in C. This made the reconstruction above relatively simple, but made (Step 3) more complicated. In the case of bipartite degree sequences (see, for example, [28] or [8]) the decomposition in (Step 2) became finer: there each vertex occurs at most once. That made the reconstruction method more complicated but it made (Step 3) much more simple and capable. To adapt this method to the UC degree sequences we cannot expect to be able to decompose into alternating cycles (bow tie!). Instead we use the following idea.

(Step $\widehat{2}$) We decompose every alternating circuit W_k into primitive alternating circuits $C_1^k, C_2^k \dots, C_{\ell_k}^k$.

This makes the reconstruction process much more demanding but provides more verifying power (as shown by the new results). For bipartite realizations, the primitive circuits are also cycles, but here we will extend this approach for UC degree sequences and will provide the analogous results.

4.1. (Step 1): parameterizing the circuit decomposition

Now let $K = (W, F \cup F')$ be a simple graph where $F \cap F' = \emptyset$ and assume that for each vertex $w \in W$ the F-degree and the F'-degree of w are the same: $d(w) = d'(w)$ for all $w \in W$. An *alternating circuit decomposition* of (F, F') is a circuit decomposition such that no two consecutive edges of any circuit are in F or in F'. By definition, that means that each circuit is of even length. Next we are going to parameterize the alternating circuit decompositions.

The set of all edges in F (in F') which are incident to a vertex w is denoted by $F(w)$ (by $F'(w)$, respectively).

If A and B are sets, denote by $[A, B]$ the complete bipartite graph with classes A and B. Let

$$\begin{aligned}\mathbb{S}(F,F') = \big\{ & s : s \text{ is a function, } dom(s) = W, \text{ and for all } w \in W \\ & s(w) \text{ is a maximum matching of the complete bipartite graph } \big[F(w), F'(w)\big]\big\}.\end{aligned} \tag{4.1}$$

Lemma 4.1. *There is a natural one-to-one correspondence between the family of all alternating circuit decompositions of (F, F') and the elements of $\mathbb{S}(F, F')$.*

Proof. If $\mathcal{C} = \{C_1, C_2, \dots, C_n\}$ is an alternating circuit decomposition of (F, F'), then define $s_{\mathcal{C}} \in \mathbb{S}(F, F')$ as follows:

$$\begin{aligned} s_{\mathcal{C}}(w) := \big\{ & \big((w,u),(w,u')\big) \in [F(w), F'(w)] : \\ & (w,u) \text{ and } (w,u') \text{ are consecutive edges in some } C_i \in \mathcal{C}\big\}. \end{aligned} \tag{4.2}$$

In turn, to each $s \in \mathbb{S}(F, F')$ assign an alternating circuit decomposition

$$\mathcal{C}_s = \{W_1^s, W_2^s \dots, W_{p_s}^s\}$$

of (F, F') as follows: Consider the bipartite graph $\mathcal{F} = \big(F, F'; R(s)\big)$, where

$$R(s) = \big\{\big((u,w),(u',w)\big) : w \in W \text{ and } \big((u,w),(u',w)\big) \in s(w)\big\}.$$

Then $\mathcal{F}$ is a 2-regular graph because for each edge $(u,v) \in F \cup F'$ there is exactly one $(u,w) \in F \cup F'$ with $\big((u,w),(u,w)\big) \in s(u)$, there is exactly one $(t,v) \in F \cup F'$ with $\big((u,v),(t,v)\big) \in s(v)$, therefore the $\mathcal{F}$-neighbors of (u,v) are (u,w) and (t,v).

Since $\mathcal{F}$ is 2-regular, it is the union of vertex disjoint cycles $\{W_i^s : i \in I\}$. Now W_i^s can also be viewed as a sequence of edges in $F \cup F'$, which is an alternating circuit in $\langle W, F \cup F' \rangle$, so $\{W_i^s : i \in I\}$ is an alternating circuit decomposition of (F, F'). Since

$$s_{\mathcal{C}_s} = s,$$

we proved the Lemma. □

If the degree sequence of both F and F' is $(d_1, \dots d_k)$, then write

$$t_{F,F'} = \prod_{i=1}^{k} (d_i!).$$

Clearly,

$$\big|\mathbb{S}\big(F,F'\big)\big| = t_{F,F'}.$$

4.2. Preparing for $\big($Step $\widehat{2}\big)$: the T-operator

For bipartite degree sequences (Step 2) simply required a cycle decomposition which was provided by the T-operator defined in Section 5.2 of [28]. For UC degree sequences the best we may hope is a decomposition into primitive circuits (See [6]). In this subsection we generalize the T-operator so that it can process any balanced red-blue graph. When this balanced red-blue graph is bipartite, the generalized and the original T-operator produce the same decomposition of alternating cycles. The new proof described in this section is simpler than that of [28] because it is described on a higher level of abstraction.

Let $[\mu] = \{1, 2, \dots, \mu\}$ be a base set, denote $S_{[\mu]}$ the symmetric group on $[\mu]$ and let Pos denote the set of **position**s, where $Pos = \{(1)^+, (2)^+, \dots, (\mu-1)^+\}$. For convenience, we consider $(i)^+ = (i+1)^-$ and allow the alternating naming $Pos = \{(2)^-, \dots, (\mu)^-\}$. Let f be a two-coloring on Pos with $f \in \{green, red\}^{Pos}$. We will describe the state of our system with the pair

$$(\pi, f) \quad : \quad \pi \in S_{[\mu]},\ f \in \{green, red\}^{Pos}.$$

Let $\mathcal{E} \subset \binom{[\mu]}{2}$ be a fixed subset which we call the set of **eligible reversals**. Assume that

$$\text{the connected components of } G = ([\mu], \mathcal{E}) \text{ are cliques.} \tag{4.3}$$

It is important to recognize that each eligible reversal consists of a pair of elements of the base set, and they do not depend on the image of those elements under π. Accordingly, to make the definitions more readable, let us define

$$\pi^{-1}(\mathcal{E}) = \Big\{\{\pi^{-1}(x), \pi^{-1}(y)\} : \{x,y\} \in \mathcal{E}\Big\}.$$

We now define an operator $T_{\mathcal{E}}$, or T for short, as the index $\mathcal{E}$ is fixed anyway. This T is a function mapping $S_{[\mu]} \times \{\mathit{green}, \mathit{red}\}^{Pos}$ into itself. To determine the image of (π, f) under T, an interval will be selected first. For that end let

$$j_{(\pi,f)} := \min\left\{ j' \in [\mu] \;\middle|\; \exists i' < j' : \; f\left((i')^+\right) = f\left((j')^-\right) = \mathit{green},\; \{i', j'\} \in \pi^{-1}(\mathcal{E}) \right\}$$

then let

$$i_{(\pi,f)} := \max\left\{ i' < j_{(\pi,f)} \;\middle|\; f\left((i')^+\right) = \mathit{green},\; \{i', j_{(\pi,f)}\} \in \pi^{-1}(\mathcal{E}) \right\}.$$

We define $\max\emptyset = -\infty$ and $\min\emptyset = +\infty$. For any integer $k : 1 \le k \le \mu$ we select two positions from Pos. Let $a_{(\pi,f)}(k) := k$ if $f((k)^-) = \mathit{green}$, and let

$$a_{(\pi,f)}(k) := \min\left\{ i' \le k \mid \forall i'' \text{ s.t. } i' \le i'' < k \;:\; f\left((i'')^+\right) = \mathit{red} \right\}$$

otherwise. Furthermore, let $b_{(\pi,f)}(k) := k$ if $f((k)^+) = \mathit{green}$, and let

$$b_{(\pi,f)}(k) := \max\left\{ j' \ge k \mid \forall j'' \text{ s.t. } k < j'' \le j' \;:\; f\left((j'')^-\right) = \mathit{red} \right\}$$

otherwise. By definition, $a_{(\pi,f)}(k) \le k \le b_{(\pi,f)}(k)$ for all $k \in [\mu]$.

We are ready now to define the image of the pair (π, f) under the operator T. If $j_{(\pi,f)} = +\infty$, then let (π, f) be a fixed point of the T operator, so its image is itself. Otherwise define $T : (\pi, f) \mapsto (\pi', f')$ as follows. For any $k \in [\mu]$, let $\overleftarrow{k}^{(\pi,f)} := a_{(\pi,f)}(k) + b_{(\pi,f)}(k) - k$. We omit the index ${}_{(\pi,f)}$ in the following.

$$\pi'(k) = \begin{cases} \pi(k) & \text{if } k \notin [a(i), b(j)], \\ \pi(\overleftarrow{a(i) + b(j) - k}) & \text{if } k \in [a(i), b(j)], \end{cases}$$

$$f'((k)^+) = \begin{cases} f((k)^+) & \text{if } 1 \le k < a(i) \text{ or } b(j) \le k < \mu, \\ \mathit{red} & \text{if } a(i) \le k < b(j). \end{cases}$$

Another description of π' is as follows: let $\left\{(a_{(\pi,f)}(i_{(\pi,f)}))^+, \ldots, (b_{(\pi,f)}(j_{(\pi,f)}))^-\right\}$ be the maximal interval (with integer endpoints) containing $\left\{(i_{(\pi,f)})^+, \ldots, (j_{(\pi,f)})^-\right\}$ such that the f-image of the new positions of the extended interval are *red*. Take a look at Figure 3. To construct π' from π, every maximal *red* interval $\{x^+, \ldots, y^-\}$ in $\{a^+, \ldots, b^-\}$ is shifted to $\{(a + b - y)^+, \ldots, (a + b - x)^-\}$, and the *green* positions within $\{a^+, \ldots, b^-\}$ are taken in reverse order in the remaining positions between the shifted *red* intervals.

Given any permutation π_0 on $[\mu]$, let $(\pi_r, f_r) = T^r(\pi_0, \mathbf{green})$, where **green** is the identically *green* function. In indices, we shorten (π_r, f_r) by writing r instead. For example, $j_r = j_{(\pi_r, f_r)}$, etc.

Lemma 4.2. *For any $r \ge 0$, the pair of endpoints of a maximal path formed by elements of $f_r^{-1}(\mathit{red})$ is an element of $\pi_r^{-1}(\mathcal{E})$.*

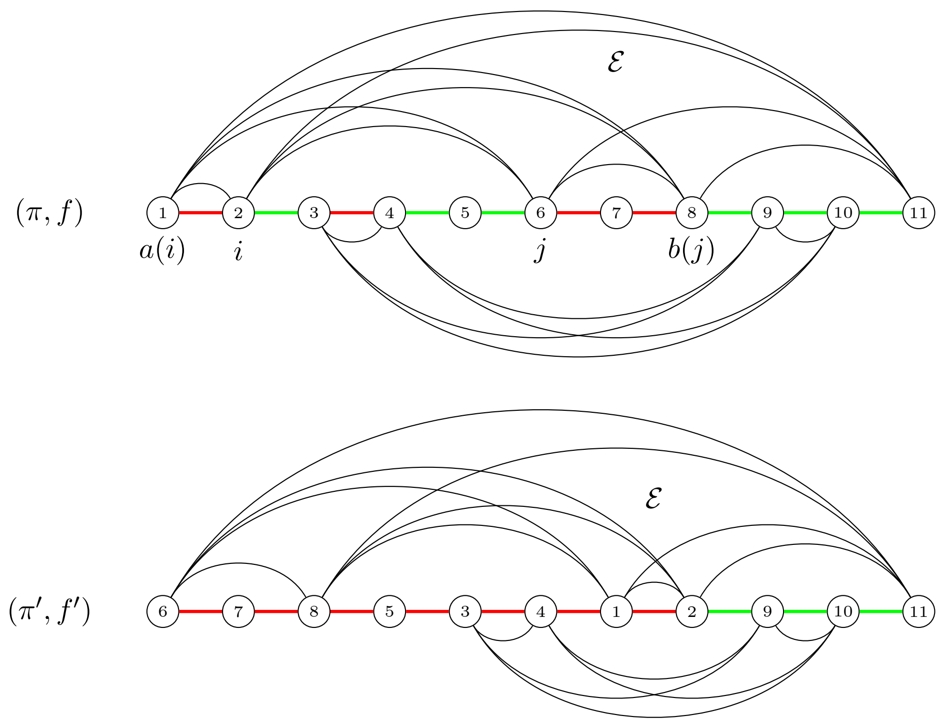

Figure 3: An example for $T(\pi, f) = (\pi', f')$. The curved arcs represent the pairs in $\mathcal{E}$. The encircled numbers are $\pi(x)$ and $\pi'(x)$, respectively, where $x = 1, \dots, 11$ from left to right (π is identity).

Proof. The statement is vacuously true when $r = 0$. Use property (4.3) and the fact that $f_r((i_r)^+) = f_r((j_r)^-) = \mathit{green}$. By induction, either $a_r(i_r) = i_r$ or $\{a_r(i_r), i_r\} \in \pi_r^{-1}(\mathcal{E})$. By definition, $\{i_r, j_r\} \in \pi_r^{-1}(\mathcal{E})$, thus we also have $\{a_r(i_r), j_r\} \in \pi_r^{-1}(\mathcal{E})$. The same argument goes through for j_r and $b_r(j_r)$. $\square$

Lemma 4.3. *The following statements hold for $r \geq 0$.*

(i) *If (π_r, f_r) is not a fixed point of the T operator then $j_r < j_{r+1}$;*

(ii) *$b_r(j_r) = j_r$;*

(iii) *$f_{r+1}((k)^+) = \mathit{green}$ for $j_r \leq k \leq \mu$;*

Proof. We proceed by induction on r. Statements (ii), (iii) are true when $r = 0$. Now suppose that $r \geq 0$ is such that statements (ii), (iii) are true for r. If (π_r, f_r) is a fixed point of T then we are done. Otherwise, since

$$f_{r+1}^{-1}(\mathit{green}) \subsetneq f_r^{-1}(\mathit{green}),$$

we have $f_r((i_{r+1})^+) = f_r((j_{r+1})^-) = \mathit{green}$. Clearly, $\{\pi_{r+1}(i_{r+1}), \pi_{r+1}(j_{r+1})\} \in \mathcal{E}$, so

$$\left\{\pi_r^{-1}(\pi_{r+1}(i_{r+1})), \pi_r^{-1}(\pi_{r+1}(j_{r+1}))\right\} \in \pi_r^{-1}(\mathcal{E}). \tag{4.4}$$

Since $f_{r+1}((j_{r+1})^-) = \mathit{green}$, we must have $j_{r+1} > b_r(j_r) = j_r$ or $j_{r+1} \le a_r(i_r)$. The first case gives $j_r < j_{r+1}$ immediately.

Next, suppose that $j_{r+1} < a_r(i_r)$. Then $\pi_{r+1}(j_{r+1}) = \pi_r(j_{r+1})$ and $\pi_{r+1}(i_{r+1}) = \pi_r(i_{r+1})$. Plugged into Equation (4.4), the definition of j_r implies that $j_{r+1} \ge j_r > a_r(i_r)$, a contradiction.

Finally, suppose that $j_{r+1} = a_r(i_r)$. Recalling that $\pi_{r+1}(a_r(i_r)) = \pi_r(j_r)$, we have

$$\left\{\pi_r^{-1}(\pi_{r+1}(i_{r+1})), \pi_r^{-1}(\pi_{r+1}(j_{r+1}))\right\} = \{i_{r+1}, j_r\}.$$

Now Equation (4.4) gives

$$\{i_{r+1}, j_r\} \in \pi_r^{-1}(\mathcal{E}),$$

and property (4.3) implies that $\{i_{r+1}, i_r\} \in \pi_r^{-1}(\mathcal{E})$. If $f_r((i_r)^-) = \mathit{green}$ then this contradicts the definition of j_r. Otherwise, $\{a_r(i_r), \ldots, i_r\}$ is a maximal *red* interval in f_r and hence, by Lemma 4.2 and property (4.3) we conclude that $\{i_{r+1}, a_r(i_r)\} \in \pi_r^{-1}(\mathcal{E})$. Again, this contradicts the choice of j_r.

Hence in all cases we conclude that $j_r < j_{r+1}$, which implies that $b_{r+1}(j_{r+1}) = j_{r+1}$ and $f_{r+2}((k)^+) = \mathit{green}$ for $j_{r+1} \le k \le \mu$. This completes the proof. ☐

Lemma 4.4. *For arbitrary π_0, $r \ge 0$, and $k \in [\mu]$, we have*

$$\pi_r(k) = \pi_0\Big(a_r(k) + b_r(k) - k\Big).$$

Proof. If $r = 1$, the statement immediately follows from the definition. Suppose the statement holds for $r-1$. If $k \notin [a_{r-1}(i_{r-1}), b_{r-1}(j_{r-1})]$ then $a_{r-1}(k) = a_r(k)$ and $b_{r-1}(k) = b_r(k)$, so

$$\pi_r(k) = \pi_{r-1}(k) = \pi_0(a_{r-1}(k) + b_{r-1}(k) - k) = \pi_0(a_r(k) + b_r(k) - k),$$

as we wished.

Suppose from now on that $k \in [a_{r-1}(i_{r-1}), b_{r-1}(j_{r-1})]$. Let

$$\ell = a_{r-1}(i_{r-1}) + b_{r-1}(j_{r-1}) - k.$$

Since the edges in $[a_{r-1}(i_{r-1}), b_{r-1}(j_{r-1})]$ are all *red* in f_r, we have

$$a_r(k) = a_{r-1}(i_{r-1}), \quad b_r(k) = b_{r-1}(j_{r-1}).$$

Writing $\overleftarrow{\ell}^{(r-1)}$ for $\overleftarrow{\ell}^{(\pi_{r-1}, f_{r-1})}$, by induction we have

$$\pi_r(k) = \pi_{r-1}\left(\overleftarrow{\ell}^{(r-1)}\right) = \pi_0\left(a_{r-1}\left(\overleftarrow{\ell}^{(r-1)}\right) + b_{r-1}\left(\overleftarrow{\ell}^{(r-1)}\right) - \overleftarrow{\ell}^{(r-1)}\right).$$

Since $a_{r-1}\left(\overleftarrow{\ell}^{(r-1)}\right) = a_{r-1}(\ell)$ and $b_{r-1}\left(\overleftarrow{\ell}^{(r-1)}\right) = b_{r-1}(\ell)$, the right hand side is equal to $\pi_0(\ell)$. Expanding it, we get

$$\pi_0(\ell) = \pi_0(a_{r-1}(i_{r-1}) + b_{r-1}(j_{r-1}) - k) = \pi_0(a_r(k) + b_r(k) - k),$$

which is what we intended to prove. □

Lemma 4.5. *If* $\{1,\mu\} \in \mathcal{E}$*, then* $\exists s \in \mathbb{N}$ *such that* $f_s^{-1}(\mathit{red}) = \mathit{Pos}$*.*

Proof. Lemma 4.2 implies that $\{1, \min\{t : f_r((t)^+) = \mathit{green}\}\} \in \pi_r^{-1}(\mathcal{E})$, therefore $\{\min\{t : f_r((t)^+) = \mathit{green}\}, \mu\} \in \pi_r^{-1}(\mathcal{E})$, except if $f_r^{-1}(\mathrm{red}) = \mathit{Pos}$ already. □

The following claim shows that we cannot have such eligible reversals $\{x, y\}$ that $x < y < j_r$ and $f((x)^+) = \mathit{green}$ and $f((y)^-) = \mathit{red}$.

Lemma 4.6. *Given* $r \geq 0$ *and any* $\{x, y\} \in \pi_r^{-1}(\mathcal{E})$ *such that* $x < y$*, either* $\{x, y\} = \{i_r, j_r\}$*, or* $f((x)^+) = \mathit{red}$*, or* $y \geq j_r + 1$*.*

Proof. The lemma trivially holds for $r = 0$. Suppose now, that $r \geq 1$.

If $f_r((x)^+) = f_r((y)^-) = \mathit{green}$, then by definition $y \geq j_r$. If $y \geq j_r + 1$, the lemma holds. If $y = j_r$, then definition of i_r implies that $x \leq i_r$. If $y = j_r$ and $x < i_r$, then $x < a_r(i_r)$. By property (4.3) and Lemma 4.2, $\{x, a_r(i_r)\} \in \pi_r^{-1}(\mathcal{E})$ holds. Since $f_r((a_r(i_r))^-) = \mathit{green}$, we have a contradiction with the definition of j_r.

Suppose, that $f_r((x)^+) = \mathit{green}$, $f_r((y)^-) = \mathit{red}$, and the lemma does not hold. By Lemma 4.3 we have $y \leq j_{r-1}$. Then we must also have $x < a_{r-1}(i_{r-1})$ (otherwise $f_r((x)^+) = \mathit{red}$, a contradiction). Thus $\pi_{r-1}^{-1}(\pi_r(x)) = x$ and $x < \pi_{r-1}^{-1}(\pi_r(y)) \leq j_{r-1}$, so $\{x, \pi_{r-1}^{-1}(\pi_r(y))\} \in \pi_{r-1}^{-1}(\mathcal{E})$. By induction, we should have $f_{r-1}((x)^+) = \mathit{red}$, which implies $f_r((x)^+) = \mathit{red}$, a contradiction.

We have checked and eliminated every possible case where the statement of the lemma is not satisfied. □

Let us define $\mathit{greenify}(\pi, f) := (\pi, \mathbf{green})$, i.e., the operator replaces the coloring f of Pos with the identical green coloring.

Theorem 4.7. $\forall r \in \mathbb{N}\ \exists w \in \mathbb{N}$ *and* $\exists g \in \{\mathit{green}, \mathit{red}\}^{\mathit{Pos}}$ *such that*

$$T^w \circ \mathit{greenify} \circ T^r(\pi_0, \mathbf{green}) = (\pi_0, g).$$

Proof. If $f_r(\mathit{Pos}) \equiv \mathit{red}$, then Lemma 4.2 implies that $\{1, \mu\} \in \pi_r^{-1}(\mathcal{E})$. Therefore Lemma 4.5 provides the existence of an appropriate s. Furthermore, π_r is the reversal of π_0, and applying T^s to $\mathit{greenify} \circ T^r(\pi_0, \mathbf{green})$ will reverse all the elements again, returning π_0 as the final permutation.

If $f_r^{-1}(red)$ is composed of multiple components, then T^s will successively work in these components. Lemma 4.4 says that the order of elements in each of these components have been reversed in π_r compared to π_0. Outside these intervals, however, π_r is identical to π_0.

Because of Lemma 4.2, we see that Lemma 4.5 implies that the maximal *red* intervals will be completely processed after a certain number of steps. Lemmas 4.3 and 4.6 together imply that if the T-operator starts working inside a component of $f_r^{-1}(red)$ then the next selected interval $[i,j]$ will also be inside until the whole component becomes *red* again. At this point, by Lemma 4.4, the order of the elements inside each *red* component have been reversed for a second time, so the final permutation equals π_0, as claimed. $\square$

Definition 4.8. The restriction of a permutation π to an interval $[\alpha,\beta]$, denoted by $\pi|_{[\alpha,\beta]}$, means that original domain $[\mu]$ of π is replaced with $[\alpha,\beta]$.

Notice that there is nothing special about the set from which π is taken. The T-operator naturally generalizes to injective maps from $[\alpha,\beta]$ to an arbitrary set.

Lemma 4.9. *Suppose that $r \in \mathbb{N}$ and $[\alpha,\beta] \subset [i_r, j_r]$ (where $\alpha < \beta$) is a proper subinterval such that any component of $f_r^{-1}(red)$ is either disjoint from $Pos[\alpha,\beta] := \{(\alpha)^+,\dots,(\beta)^-\}$ or entirely contained by it. If $f_r((\alpha)^+) = f_r((\beta)^-) = green$, then*

$$T(\pi_r|_{[\alpha,\beta]}, f_r|_{Pos[\alpha,\beta]}) = (\pi_r|_{[\alpha,\beta]}, f_r|_{Pos[\alpha,\beta]}) = T^r(\vartheta, \mathbf{green}),$$

where $\vartheta = \pi_0|_{(\alpha,\beta)} \cup \{(\alpha, \pi_r(\alpha)), (\beta, \pi_r(\beta))\}$.

Proof. Both $\pi_0|_{[\alpha,\beta]}, \pi_r|_{[\alpha,\beta]}$ map $[\alpha,\beta]$ to the same set, because of the assumption on the components of $f_r^{-1}(red)$. Our other assumptions imply that $\{\alpha,\beta\} \notin \mathcal{E}$ and that $(\pi_r|_{[\alpha,\beta]}|f_r|_{[\alpha,\beta]})$ is a fixed point of the T-operator. Lemma 4.2 guarantees that for $k \le r$, either $\alpha < i_{(\pi_k,f_k)} < j_{(\pi_k,f_k)} < \beta$, or $[i_{(\pi_k,f_k)}, j_{(\pi_k,f_k)}] \cap (\alpha,\beta) = \emptyset$. In the former case, the T-operator and the restriction operation trivially commute:

$$T(\pi_k|_{[\alpha,\beta]}, f_k|_{[\alpha,\beta]}) = T(\pi_k, f_k)|_{[\alpha,\beta]}.$$

(It is necessary to define $\vartheta(\alpha)$ and $\vartheta(\beta)$ separately due to the possibility that $j_{(\pi_k,f_k)} = \alpha$ or $i_{(\pi_k,f_k)} = \beta$ for some k.) $\square$

4.3. (Step $\widehat{2}$) — decomposing a circuit into primitive circuits

Given $X, Y \in \mathbb{G}$ (we do not specify whether the degree sequence is unconstrained or bipartite), and $s \in S_{X,Y}$, we construct a path between X and Y in $\mathbb{G}(d)$ as follows. The matching s decomposes $\nabla = X \triangle Y$ into alternating circuits

$$W_1, \dots, W_{p_s}. \tag{4.5}$$

We will use our operator T to decompose the circuits above into primitive circuits, and show that this gives us the ability to reconstruct from a given transition (with some extra parameters) the realization X, Y and the original matching function s at any given moment.

Let W_k be an arbitrary but fixed circuit from the collection (4.5). Let v_1v_2 be the lexicographically first edge of W_k, where v_1 precedes v_2. Let the Eulerian trail induced by s starting on the edge v_1v_2 be $v_1v_2v_3v_4\dots v_{|E(W_k)|}v_{|E(W_k)|+1}$, where $v_{|E(W_k)|+1} = v_1$. Let $\mu = |E(W_k)|+1$, $\pi_0 = \mathrm{id}_{[\mu]}$, $f_0 = \textbf{green}$, and

$$\mathcal{E} = \left\{ \{x,y\} \in \binom{[\mu]}{2} \ : \ v_x = v_y \text{ and } x \equiv y \pmod 2 \right\}. \tag{4.6}$$

By transitivity, this set possesses property (4.3), so we can apply the T-operator on π_0 with $\mathcal{E}$ as the set of eligible reversals. For all $r \in \mathbb{N}$ let $(\pi_r, f_r) = T^r(\pi_0, f_0)$.

Lemma 4.10. *Given $r \in \mathbb{N}$, visiting the vertices $v_{\pi_r(1)}v_{\pi_r(2)}\dots v_{\pi_r(|E(W_k)|+1)}$ (in this order) gives an Eulerian trail of W_k in the graph (V, ∇).*

Proof. Easily seen by induction on r. Lemma 4.2 and the definition of the T operator implies that we get π_r by reversing some intervals of the trail defined by π_{r-1} whose first and last vertices are identical. Consequently, every edge is visited by the new trail too. ☐

It is also clear, by definition, that $(x)^+$ in the set *Pos* coincides with the xth edge along the Eulerian trail π_r in the graph (V, ∇). We will denote this edge by $e_x^{(r)} := v_{\pi_r(x)}v_{\pi_r(x+1)}$.

Lemma 4.11. *For $k = 1, \dots, p_s$ and $r \in \mathbb{N}$ let*

$$\begin{aligned} E_r^k &= E(X) \triangle \left(\cup_{i=1}^{k-1} E(W_i) \right) \triangle \left\{ e_x^{(r)} \ : \ (x)^+ \in f_r^{-1}(red) \right\}, \\ G_r^k &= (V, E_r^k). \end{aligned}$$

Then the graph G_r^k is a realization from $\mathbb{G}$ and the closed vertex sequence π_r describes an alternating circuit in it.

In words, we obtain G_r^k from X by exchanging edges with non-edges (and vice versa) in the following subsets of edges: any W_i for $1 \le i \le k-1$ and any edges in W_k which are *red* in f_r. For any k, r we call G_r^k a **milestone**. Milestones are special realizations: both $G_r^k \triangle X$ and $G_r^k \triangle Y$ are subgraphs of $X \triangle Y$. Milestones are uniquely determined by (X, Y, s) and the fixed lexicographical order.

Lemma 4.12. *For any $r \in \mathbb{N}$, we can describe $\pi_r^{-1}(\mathcal{E})$ as the set of endpoints of even circuits formed by subintervals of the Eulerian trail in (V, E_r^k) defined by π_r:*

$$\pi_r^{-1}(\mathcal{E}) = \left\{ \{x,y\} \in \binom{[\mu]}{2} \ : \ v_{\pi_r(x)} = v_{\pi_r(y)} \text{ and } x \equiv y \pmod 2 \right\}.$$

Proof. From (4.6), we have

$$\begin{aligned}\pi_r^{-1}(\mathcal{E}) &= \left\{ \{\pi_r^{-1}(x), \pi_r^{-1}(y)\} \;:\; \{x,y\} \in \binom{[\mu]}{2},\ v_x = v_y \text{ and } x \equiv y \pmod 2 \right\} \\ &= \left\{ \{x,y\} \in \binom{[\mu]}{2} \;:\; v_{\pi_r(x)} = v_{\pi_r(y)} \text{ and } \pi_r(x) \equiv \pi_r(y) \pmod 2 \right\},\end{aligned}$$

because π_r is a permutation. It is enough to show that π_r preserves parity, i.e., $\pi_r(x) \equiv x \pmod 2$ for any x. For $r = 0$ this is trivial. Suppose π_{r-1} preserves parity. For $x \notin [a_{r-1}(i_{r-1}), b_{r-1}(j_{r-1})]$, we have $\pi_r(x) \equiv \pi_{r-1}(x) \pmod 2$, so parity is preserved.

Now suppose that $x \in [a_{r-1}(i_{r-1}), b_{r-1}(j_{r-1})]$. First observe that Lemma 4.2 implies that $a_{r-1}(y) \equiv b_{r-1}(y) \pmod 2$ for arbitrary y. Hence $a_{r-1}(i_{r-1}) \equiv i_{r-1} \pmod 2$, and since $\{i_{r-1}, j_{r-1}\} \in \pi_{r-1}(\mathcal{E})$ we know that $i_{r-1} \equiv j_{r-1}$. As $j_{r-1} = b_{r-1}(j_{r-1})$ by Lemma 4.3, it follows that $a_{r-1}(i_{r-1}) \equiv b_{r-1}(j_{r-1}) \pmod 2$. Now

$$\pi_r(x) = \pi_{r-1}\big(a_{r-1}(z) + b_{r-1}(z) - z\big)$$

where

$$z = a_{r-1}(i_{r-1}) + b_{r-1}(j_{r-1}) - x \;\equiv\; x \pmod 2.$$

Hence, by induction,

$$\pi_r(x) \equiv a_{r-1}(z) + b_{r-1}(z) - z \;\equiv\; z \;\equiv\; x \pmod 2,$$

completing the proof. □

Denote by ℓ_k the maximum ℓ for which $\pi_{\ell-1} \neq \pi_\ell$. For any $1 \le r \le \ell_k$, let

$$E(C_r^k) = \left\{ e_x^{(r-1)} \;:(x)^+ \in f_r^{-1}(\mathit{red}) \setminus f_{r-1}^{-1}(\mathit{red}) \right\} = \tag{4.7}$$

$$= \left\{ e_x^{(r-1)} \;:\; (x \in [i_{r-1}, j_{r-1} - 1]) \wedge \big((x)^+ \in f_{r-1}^{-1}(\mathit{green})\big) \right\}. \tag{4.8}$$

Indeed, we could replace $e_x^{(r-1)}$ with $e_x^{(0)}$ in the above definition, as the endpoints of *green* edges are still in their original position. Take the list of edges of W_k starting with v_1v_2 in the order defined by the Eulerian trail $\pi_0 = \mathrm{id}_{[\mu]}$. This order can be restricted to the edges of C_r^k, so there is a natural Eulerian trail on C_r^k, too.

Lemma 4.13. *The sequence $C_1^k, \dots C_{\ell_k}^k$ is a primitive alternating circuit decomposition of W_k.*

Proof. Observe that f_r defines a coloring of the edges of W_k: the edge $v_{\pi_r(\ell)}v_{\pi_r(\ell+1)}$ has color $f_r((\ell)^+)$. Moreover, as r increases, *red* edges stay *red*. As W_k is an alternating circuit, $|E(W_k)|$ is divisible by two, so $\{1, |E(W_k)|+1\} \in \mathcal{E}$. Lemma 4.5

implies that $\bigcup_{r=1}^{\ell_k} C_r^k$ is indeed an edge-disjoint partition of W_k. Furthermore, C_r^k is an alternating circuit by Lemma 4.2 and the definition of $\mathcal{E}$.

Suppose C_r^k visits some vertex three times, that is

$$\exists x < y < z \text{ such that } i_{r-1} \leq x, y, z < j_{r-1} \text{ and } v_{\pi_r(x)} = v_{\pi_r(y)} = v_{\pi_r(z)}.$$

The proof of Lemma 4.12 shows that π_r preserves parity. If $x \equiv y \pmod 2$ then $y \geq j_{r-1}$, a contradiction. Similarly, we must have $y \not\equiv z \pmod 2$ and $x \not\equiv z \pmod 2$. In any case, we have a contradiction.

Similarly, if an even number of steps lead from one copy of a vertex to another copy of it on C_r^k, then j_{r-1} is not minimal, contradiction. □

Observe, that a primitive alternating circuit on a bipartite graph is a cycle. The decomposition produced by the above process in the bipartite case is identical to the one described in [28].

4.4. (Step 3) — Describing the switch sequence along a primitive alternating circuit

In this subsection we will construct switch sequences to transform one milestone realization into the next one, using Algorithm 2.1. Recursive application of this procedure will provide the entire switch sequence between realizations X and Y.

From Lemma 4.11 and Lemma 4.13, it follows that for any $1 \leq k \leq p$ and $1 \leq r \leq \ell_k$ (where $p = p_s$) we have

$$E_r^k = E_{r-1}^k \triangle E(C_r^k).$$

Clearly, $X = G_0^1$ and $Y = G_{\ell_p}^p$. Also, $G_{\ell_k}^k = G_0^{k+1}$ for $1 \leq k < p$.

We have

$$E(C_r^k) = \{v_x v_{x+1} \ : \ i_{r-1} \leq x < j_{r-1} \text{ and } f_{r-1}((x)^+) = \mathit{green}\},$$

using the observation made below (4.8), that π_0 matches π_{r-1} on C_r^k.

$$\text{Let } x_1 := v_y \in V(C_r^k) \text{ be a vertex of minimum degree in } G_{r-1}^k[V(C_r^k)], \\ \text{such that } y \text{ is minimal wrt. to this condition.} \tag{4.9}$$

We take the switch sequence between G_{r-1}^k and G_r^k which is produced by Algorithm 2.1 as described by Lemma 2.5. The symmetric difference of G_{r-1}^k and G_r^k is $E(C_r^k)$. To apply Lemma 2.5, we label the vertices of C_r^k by $x_1, x_2, \ldots$ following the natural Eulerian circuit on C_r^k (either forwards or in reverse), starting with a non-edge $x_1 x_2 \notin E_{r-1}^k$.

4.5. Reconstructing the switch sequence

Suppose that Z is a realization which lies on the path from X to Y with respect to some $s \in S_{X,Y}$. Recall Lemma 2.6 and Equation (2.5), which in our setting translates to

$$R = \big((Z \triangle G^k_{r-1}) \setminus E(C^k_r)\big) \cup Q, \tag{4.10}$$

where either $Q = \emptyset$ or Q contains exactly one edge of C^k_r. Unfortunately, R may intersect W_i for some $i \neq k$, so it is favorable to work on a graph Z' which is close to Z:

$$Z' = Z \triangle R. \tag{4.11}$$

Although generally Z' is not a realization of $\mathbf{d}$, it is well-behaved with respect to alternating circuits other than W_k. From the definitions and Lemma 2.6(f) we can read off that

$$\begin{aligned}
&Z' \triangle X,\ Z' \triangle Y \ \subseteq E(C^k_r) \subseteq X \triangle Y,\\
&E(Z') \cap E(W_i) = E(X) \cap E(W_i) \ \text{ for } \ i > k,\\
&E(Z') \cap E(W_i) = E(Y) \cap E(W_i) \ \text{ for } \ i < k,\\
&E(Z') \cap E(C^k_j) = E(X) \cap E(C^k_j) \ \text{ for } \ j > r,\\
&E(Z') \cap E(C^k_j) = E(Y) \cap E(C^k_j) \ \text{ for } \ j < r.
\end{aligned} \tag{4.12}$$

If Z is a **milestone** (that is, $Z = G^k_r$ for some k, r) then $R = \emptyset$ and we have $E(Z') \cap E(C^k_r) = E(Y) \cap E(C^k_r)$. The Eulerian trail associated to π_r is alternating in $Z = G^k_r$.

If Z is an **intermediate realization** strictly between G^k_{r-1} and G^k_r, then there may not exist an Eulerian trail on W_k which is alternating in $Z' = Z \Delta R$. For W_i ($i \neq k$), the trail induced by $s \in S_{X,Y}$ on W_i is alternating in Z' (but generally it is not alternating in Z). Our next goal is to slightly modify π_{r-1}, such that it induces a trail in Z' which is alternating between edges and non-edges with a constant number of exceptions.

According to Lemma 2.6(f), the symmetric difference of Z' and G^k_{r-1} are two intervals of edges on C^k_r. Without loss of generality,

$$Z' \triangle G^k_{r-1} = \{v_x v_{x+1} \ : \ x \in \mathcal{I} \text{ and } f_{r-1}((x)^+) = \textit{green}\} \tag{4.13}$$

where $\mathcal{I} = [\alpha, \beta) \cup [\gamma, \delta)$ or $\mathcal{I} = [i_{r-1}, \alpha) \cup [\beta, \gamma) \cup [\delta, j_{r-1})$, such that $i_{r-1} \leq \alpha \leq \beta \leq \gamma \leq \delta \leq j_{r-1}$ and $\alpha, \beta, \gamma, \delta$ are chosen in such a way that $\mathcal{I}$ is minimal. Let us define $\pi_{Z'}$ as follows ($\pi_{Z'}$ depends on X, Y, and s, too). If $\mathcal{I} = [\alpha, \beta) \cup [\gamma, \delta)$, then let

$$\pi_{Z'}(x) = \begin{cases} \pi_{r-1}(\overleftarrow{x}^{(r-1)}), & \text{if } x \in [\alpha, \beta] \cup [\gamma, \delta] \\ \pi_{r-1}(x), & \text{otherwise.} \end{cases} \tag{4.14}$$

In other words, π_{r-1} is reversed on the maximal f_{r-1}-*red* intervals of $[\alpha,\beta]\cup[\gamma,\delta]$.

If $\mathcal{I}$ cannot match the previous form, then we must have $\mathcal{I}=[i_{r-1},\alpha)\cup[\beta,\gamma)\cup[\delta,j_{r-1})$, such that $i_{r-1}<\alpha$ and $\delta<j_{r-1}$. In this case, let

$$\pi_{Z'}(x)=\begin{cases}\pi_{r-1}\left(\overleftarrow{(i_{r-1}+j_{r-1}-x)}^{(r-1)}\right), & \text{if } x\in[i_{r-1},\alpha]\cup[\beta,\gamma]\cup[\delta,j_{r-1}]\\ \pi_{r-1}(i_{r-1}+j_{r-1}-x), & \text{otherwise.}\end{cases} \tag{4.15}$$

Lemma 4.14. *The Eulerian trail defined by $\pi_{Z'}$ on W_k alternates on Z' with the exception of at most 4 pairs of chords.*

Proof. Since π_{r-1} alternates on G^k_{r-1}, it follows that π_{r-1} also alternates at $v_{\pi_{r-1}(x)}$ in Z' for $x\notin\mathcal{I}$. For any $x\in\mathcal{I}\setminus\{\alpha,\beta,\gamma,\delta\}$, π_{r-1} alternates at $v_{\pi_{r-1}(x)}$ in Z' if $f_{r-1}((x)^-)=f_{r-1}((x)^+)$. To take care of alternation at $v_{\pi_{r-1}(x)}$ where $f_{r-1}((x)^-)\neq f_{r-1}((x)^+)$ and $x\in\mathcal{I}\setminus\{\alpha,\beta,\gamma,\delta\}$ we have to reverse these two subtrails on the edges which also belong to $f^{-1}_{r-1}(red)$. In addition, if $\mathcal{I}=[i_{r-1},\alpha)\cup[\beta,\gamma)\cup[\delta,j_{r-1})$ such that $i_{r-1}<\alpha$ and $\delta<j_{r-1}$, then we reverse the trail on the whole circuit C^k_r, so that the trail defined by $\pi_{Z'}$ alternates at $v_{\pi_{Z'}(i_{r-1})}$ and $v_{\pi_{Z'}(j_{r-1})}$ in Z'. Overall there are at most 4 non-alternations of the trail associated to $\pi_{Z'}$ in Z', which occur at a subset of $\{v_{\pi_{Z'}(\alpha)},v_{\pi_{Z'}(\beta)},v_{\pi_{Z'}(\gamma)},v_{\pi_{Z'}(\delta)}\}$. ☐

Definition 4.15. Let $s(X,Y,Z)\in\mathbb{S}(\nabla\cap Z',\nabla\setminus Z')$ be the set of matchings we obtain by modifying the original $s\in S_{X,Y}$ such that it traces $\pi_{Z'}$ on W_k in Z' (see Lemma 4.10). The non-alternating pairs are not stored in $s(X,Y,Z)$.

Lemma 4.16. $|\mathbb{S}(\nabla\cap Z',\nabla\setminus Z')|\leq n^4\cdot|S_{X,Y}|$.

Proof. According to Lemma 4.14, the sites of non-alternation of $\pi_{Z'}$ in Z' are vertices of certain edges in R (x_1 and the other ends of the start-, end-, and current-chord). At each of these sites, $\deg_{\nabla\cap Z'}(v)=\frac{1}{2}\deg_\nabla(v)+1$ and $\deg_{\nabla\setminus Z'}(v)=\frac{1}{2}\deg_\nabla(v)-1$, or the other way around. Recall Lemma 4.1. The number of maximum matchings between edges incident to v in $\nabla\cap Z'$ and in $\nabla\setminus Z'$ is $\left[(\frac{1}{2}\deg_\nabla(v)+1)!\right]$. Thus there is an extra factor of $\frac{1}{2}\deg_\nabla(v)+1\leq n$ compared to the respective factor in the enumeration of $S_{X,Y}=\mathbb{S}(X\setminus Y,Y\setminus X)$. ☐

Lemma 4.17. *Given Z, R, ∇, $s(X,Y,Z)$, and w given by Theorem 4.7 for π_{r-1}, a constant number of bits σ are sufficient to uniquely determine X, Y, and s.*

Proof. The matching $s(X,Y,Z)$ assembles W_i and the trails given by s on them for $i\neq k$ (because the lexicographical order is used to number W_i). The edges $E(W_k)$ are assembled into at most three trails.

If $E(W_k)$ is assembled into one trail by $s(X,Y,Z)$, then $Z'=Z\triangle R=G^k_{r-1}$ and thus $\pi_{Z'}=\pi_{r-1}$. A single bit in σ is dedicated to indicate which of the two directions the trail of π_{r-1} starts from v_1. We can reconstruct π_0 from π_{r-1} and

w via Theorem 4.7. The T-operator reproduces the primitive alternating circuits C_j^k for $1 \leq j \leq \ell_k$ (Lemma 4.13). Clearly, $X = G_{r-1}^k \triangle \cup_{i=1}^{k-1} W_i \triangle \cup_{j=1}^{r-1} C_j^k$ and $Y = X \triangle \nabla$.

If $E(W_k)$ is not assembled into one trail by $s(X,Y,Z)$, the reconstruction is trickier. Let σ contain the list of assembly instructions of the at most three trails (the original trail s is a closed trail). The non-alternations in $s(X,Y,Z)$ occur precisely at the boundaries of intervals of positions corresponding to $\mathcal{I}$. A flag in σ is dedicated to signaling whether Equation (4.14) or 4.15 holds for $\pi_{Z'}$.

Suppose first, that Equation (4.14) holds. Then $\mathcal{I} = [\alpha,\beta) \cup [\gamma,\delta)$, and the values of $\alpha,\beta,\gamma,\delta$ are known (these are the sites of non-alternation). Apply the T-operator repeatedly to the positions in the interval $[\alpha,\beta)$ (i.e., restrict the T-operator to this interval), starting from an identically *green* coloring until a fixed point of the T-operator is reached. According to Theorem 4.9, this transforms $\pi_{Z'}|_{[\alpha,\beta]}$ back to $\pi_{r-1}|_{[\alpha,\beta]}$. Repeat the procedure for the $[\gamma,\delta)$ interval, too. As in the case when Z is a milestone, π_{r-1} and w determine π_0, which in turn determines G_{r-1}^k, and then X and Y.

Lastly, if Equation (4.15) holds, then $\mathcal{I} = [i_{r-1},\alpha) \cup [\beta,\gamma) \cup [\delta, j_{r-1})$, such that $i_{r-1} < \alpha$ and $\delta < j_{r-1}$. Compared to the previous case, the extra complexity is determining i_{r-1} and j_{r-1} before we could determine π_{r-1}. Because $f_{r-1}((\alpha)^+) = f_{r-1}((\delta)^-) = \mathit{green}$, we have

$$
\begin{aligned}
j_{r-1} &= \min\{x > \delta \;:\; \exists y < \alpha \quad \text{such that} \quad (v_x = v_y) \wedge \big(x \equiv y \pmod 2\big)\},\\
i_{r-1} &= \max\{y < \alpha \;:\; (v_{j_{r-1}} = v_y) \wedge \big(j_{r-1} \equiv y \pmod 2\big)\}.
\end{aligned}
$$

□

Let us define a list of additional parameters:

$$
\begin{aligned}
B(X,Y,Z,s) &:= (x_1, \sigma, R, w),\\
\mathbb{B} &= \Big\{ B(X,Y,Z,s) \;:\; Z \in \Upsilon(X,Y,s) \Big\}.
\end{aligned}
\tag{4.16}
$$

Lemma 4.18. *The cardinality of the parameter set $\mathbb{B}$ is $\mathcal{O}(n^8)$ in the unconstrained case, and $|\mathbb{B}| = \mathcal{O}(n^6)$ in the bipartite case.*

Proof. Clearly $0 \leq w \leq n^2$, because the generalized T-operator decreases the number of *green* positions in each iteration. Lemma 2.6(g) claims that given x_1, R has only $\mathcal{O}(n^5)$ possible values ($\mathcal{O}(n^3)$ in the bipartite case). Since σ has a description of constant size, and there are at most n choices for x_1, we have $|\mathbb{B}| \leq \mathcal{O}(n^8)$ (and $\mathcal{O}(n^6)$ in the bipartite case). □

Let us also define the auxiliary structure:

$$
\begin{aligned}
\widehat{M}(X,Y,Z) &:= A_X + A_Y - A_Z,\\
\mathcal{M} &= \Big\{ \widehat{M}(X,Y,Z) \;:\; Z \in \Upsilon(X,Y,s) \Big\}.
\end{aligned}
\tag{4.17}
$$

where A_X, A_Y, A_Z are the adjacency matrices of X, Y, Z, respectively. (In the adjacency matrices in the columns the vertices are enumerated from left to right, while in the rows from top to bottom. Therefore the matrix is symmetric with identically zero main diagonal.) In this subsection we focus on the role of $\widehat{M}$ in the reconstruction process. We will further discuss its properties in Section 6.

Lemma 4.19. *The function*

$$\Phi_Z(X, Y, s) = (B(X, Y, Z, s), \widehat{M}(X, Y, Z), s(X, Y, Z))$$

is injective on the tuples satisfying $Z \in \Upsilon(X, Y, s)$. In other words, Φ_Z has an inverse function

$$\Psi_Z : \mathcal{M} \times \mathbb{B} \times \bigcup_{Z', \nabla} \mathbb{S}(\nabla \cap Z', \nabla \setminus Z') \to \{(X, Y, s) : Z \in \Upsilon(X, Y, s)\}$$

(naturally, Ψ_Z is only defined on the image set of Φ_Z).

Proof. The graph $\nabla = X \triangle Y$ is determined by Z and $\widehat{M}$ (but these data alone do not separate the X-edges and Y-edges). The graph Z' is determined by $B(X, Y, Z, s)$ and Z. Lemma 4.17 claims that (X, Y, s) can be reconstructed from these objects. □

5. Directed degree sequences

Now it is time to extend our management for directed degree sequences. This short description goes more or less in parallel with [11].

Let $\vec{G}$ be a simple directed graph (parallel edges and loops are forbidden, but oppositely directed edges between two vertices are allowed) with vertex set $X(\vec{G}) = \{x_1, x_2, \ldots, x_n\}$ and edge set $E(\vec{G})$. For every vertex $x_i \in X$ we associate two numbers: the *out-degree* and the *in-degree* of x_i. These numbers form the directed degree bi-sequence $\vec{\mathbf{d}} = (\vec{\mathbf{d}}_{\text{out}}, \vec{\mathbf{d}}_{\text{in}})$.

We introduce the following *bipartite representation* of $\vec{G}$: let $B(\vec{G}) = (U, V; E)$ be a bipartite graph where each class consists of one copy of every vertex from $X(\vec{G})$. The edges adjacent to a vertex u_x in class U represent the out-edges from x, while the edges adjacent to a vertex v_x in class V represent the in-edges to x (so a directed edge xy corresponds the edge $u_x v_y$). If a vertex has zero in- (respectively out-) degree in $\vec{G}$, then we delete the corresponding vertex from $B(\vec{G})$. (This representation was used by Gale [12], but one can find it already in [30].) The directed degree bi-sequence $\vec{\mathbf{d}}$ gives rise to a bipartite degree sequence $\vec{\mathbf{D}}$:

$$\vec{\mathbf{D}} = \big((\vec{\mathbf{d}}_{\text{out}}(x_1), \ldots, \vec{\mathbf{d}}_{\text{out}}(x_n)), (\vec{\mathbf{d}}_{\text{in}}(x_1), \ldots, \vec{\mathbf{d}}_{\text{in}}(x_n))\big).$$

Since there are no loops in our directed graph, there cannot be any (u_x, v_x) edge in its bipartite representation — these vertex pairs are **non-chords**. It is easy to see that these forbidden edges form a forbidden (partial) matching $\mathcal{F}$ in the bipartite graph $B(\vec{G})$, or in more general terms, in $B(\vec{\mathbf{D}})$, and we call this a **restricted** bipartite degree sequence.

Definition 5.1. For restricted bipartite degree sequences, the set of chords is the vertex pairs of the form $u_x v_y$ where $x \neq y$.

By definition, $\mathbb{G}(\vec{\mathbf{D}})$ is the set of all bipartite realizations of $\vec{\mathbf{D}}$ which avoid the non-chords from $\mathcal{F}$. Now it is easy to see that the bipartite graphs in $\mathbb{G}(\vec{\mathbf{D}})$ are in one-to-one correspondence with the possible realizations of the directed degree bi-sequence.

Consider two oppositely oriented triangles, $\overrightarrow{C_3}$ and $\overleftarrow{C_3}$. In order to move between these two realizations of the same degree sequence, Kleitman and Wang [26] observed that a new operation is needed, and they proved that with this extra "switching" operation the space of realizations becomes irreducible. Take the symmetric difference ∇ of the bipartite representations $B(\overrightarrow{C_3})$ and $B(\overleftarrow{C_3})$. It contains exactly one alternating cycle (the edges come alternately from $B(\overrightarrow{C_3})$ and $B(\overleftarrow{C_3})$), s.t. each vertex pair of distance 3 along the cycle in ∇ is a non-chord. In this alternating cycle no "classical" switch can be performed. To address this issue, we need an extra switch operation, which is the bipartite analogue of the operation introduced by Kleitman and Wang: we exchange all edges coming from $B(\overrightarrow{C_3})$ with all edges coming from $B(\overleftarrow{C_3})$ in one operation.

In general, a **triple-switch** is defined as follows: take a length-6 alternating cycle C in ∇, and if one of the three vertex pairs of distance 3 in C forms a non-chord, we exchange all edges of C to non-edges and vice versa. It is a well-known fact ([6], [8]) that the set $\mathbb{G}(B(\vec{\mathbf{D}}))$ of all realizations is irreducible under switches and triple-switches that avoid the $\mathcal{F}$-edges.

The example of $\overrightarrow{C_3}$ and $\overleftarrow{C_3}$ demonstrates why the triple-switch operation is necessary. However, as long as some steps of the Markov-chain require choosing 6 vertices, it seems wasteful to not perform the triple-switch simply because some of the vertex pairs of distance 3 are chords.

In this paper, we relax the restrictions on triple-switches: given a length-6 alternating cycle C in ∇, a triple switch is valid if and only if at least one of the three vertex pairs of distance 3 in C is a non-chord. This relaxation allows us to shave off a factor of n^4 from the mixing time of the Markov chain. To see this, compare the proofs of Theorem 7.3 and Theorem 7.8.

The inner loop of Algorithm 2.1 has to be modified, because the conclusion of Lemma 2.4 does not necessarily hold in the directed case. The adaptation of Sweep in Algorithm 5.1 works on the bipartite representation $B(\vec{G})$ instead of the directed graph $\vec{G}$. If Z_q gets its value from Triple-switch, then Lemma 2.6(d) applies to it, otherwise Lemma 2.6(c) holds for Z_q. Because of this, the statements

Algorithm 5.1 Sweeping a primitive circuit in the bipartite representation. The DIRECTED SWEEP assumes that $x_1x_2 \notin E(G)$ and that $(x_1, x_2, \dots, x_{2\ell})$ is a primitive alternating circuit.

```
function TRIPLE-SWITCH(G, x_1, [x_{2t-2}, x_{2t-1}, x_{2t}, x_{2t+1}, x_{2t+2}])
    return G + {x_1 x_{2t-2}, x_{2t-1} x_{2t}, x_{2t+1} x_{2t+2}} − {x_{2t-2} x_{2t-1}, x_{2t} x_{2t+1}, x_{2t+2} x_1}
end function

procedure DIRECTED SWEEP(G, [x_1, x_2, ..., x_{2ℓ}]) → [Z_1, Z_2, ..., Z_{ℓ-2}, ⟨Z_{ℓ-1}⟩]
    Z_0 ← G
    q ← 1
    end ← 2
    L ← {2i ∈ 2N : 4 ≤ 2i ≤ 2ℓ, x_1 x_{2i} is a chord and x_1 x_{2i} ∈ E(G)}

    while end < 2ℓ do
        start ← min{2i ∈ L : 2i > end}
        2t ← start − 2

        while 2t ≥ end do
            if x_1 x_{2t} is a non-chord then
                Z_q ← TRIPLE SWITCH(Z_{q-1}, x_1, [x_{2t-2}, ..., x_{2t+2}])
                2t ← 2t − 2
            else if x_1 x_{2t} is a chord then
                Z_q ← Z_{q-1} − {x_1 x_{2t+2}, x_{2t} x_{2t+1}} + {x_1 x_{2t}, x_{2t+1} x_{2t+2}}
            end if
            q ← q + 1
            2t ← 2t − 2
        end while

        end ← start
    end while

end procedure
```

of Lemma 2.6(f) and (g), and Lemma 4.19 about the bipartite case apply to the directed case as well.

We are ready to define our switch Markov chain on $(\mathbb{G}(\vec{\mathbf{D}}), P)$ for the restricted bipartite degree sequence $\vec{\mathbf{D}}$. The transition (probability) matrix P of the Markov chain is defined as follows: let the current realization be G. Then

(a) with probability $1/2$ we uniformly choose a set of two vertices $\{u, u'\}$ from U and a set of two vertices $\{v, v'\}$ from V. There are two matchings, $\{uv, u'v'\}$ and $\{uv', u'v\}\}$, between these sets. Let F be one of these matchings, chosen randomly, and let F' be the other matching. If both F and F' consist of chords only and $F \subseteq E(G)$ and $F' \cap E(G) = \emptyset$, then perform the switch (so $E(G') = (E(G) \cup F') \setminus F)$, otherwise $G' = G$.

(b) With probability $1/2$ we choose a set of three vertices from U and a set of three vertices from V. Let F and F' be a uniformly randomly selected pair of disjoint perfect matchings between these sets. If both F and F' consist of chords only, and the remaining matching between the two chosen sets contains a non-chord, and $F \subseteq E(G)$ and $F' \cap E(G) = \emptyset$, then perform the triple-switch (so $E(G') = E(G) \cup F' \setminus F)$, otherwise $G' = G$.

The (triple-)switch moving from G to G' is unique, therefore the probability of this transformation (the *jumping probability* from G to $G' \neq G$) is:

$$\mathrm{Prob}(G \to_{(a)} G') := P(G, G') = \frac{1}{4} \cdot \frac{1}{\binom{|U|}{2}\binom{|V|}{2}}, \tag{5.1}$$

and

$$\mathrm{Prob}(G \to_{(b)} G') := P(G, G') = \frac{1}{24} \cdot \frac{1}{\binom{|U|}{3}\binom{|V|}{3}}. \tag{5.2}$$

The probability of transforming G to G' (or vice versa) is time-independent and symmetric. Therefore, P is a symmetric matrix, where the entries in the main diagonal are non-zero, but (possibly) distinct values. Again, $P(G, G) \geq \frac{1}{2}$, because if (F, F') corresponds to a feasible (triple-)switch, then (F', F) does not. Therefore the chain is aperiodic and the eigenvalues of its transition matrix are non-negative.

However it is important to recognize that in papers [16] and [18] there was a slightly different Markov chain studied, where it is assumed that the degree sequences under study are irreducible using switches only. One notable example is the regular directed degree sequence. Papers [2] and [27] provide a full characterization of directed degree sequences with this property.

6. The auxiliary matrix $\widehat{M}$

The auxiliary matrix $\widehat{M} = A_X + A_Y - A_Z$ is a linear combination of three adjacency matrices. The row and columns sums are equal to the corresponding degrees prescribed by $\mathbf{d}$. If $Z = G_r^k$, then $G_r^k \triangle X \subseteq X \triangle Y$ implies that $\widehat{M}$ is a 0–1 matrix. If Z is an intermediate realization, $\widehat{M}$ is still a 0–1 matrix except on the entries associated to edges in R, since $(Z \triangle R) \triangle X \subseteq X \triangle Y$. These $+2$ and -1 entries will be called **bad entries**, and the chords to which they correspond to are called type-(2) and type-(-1) chords, respectively.

Lemma 6.1. *If R falls under case* (c) *or* (d) *of Lemma 2.6, then R contains at most two type-*(2) *and at most one type-*(-1) *chords.*

Proof. Lemma 2.6(c) or (d) claims that R has at most three elements. Of these, $x_1 x_{start}$ and $x_1 x_{end}$ are edges in X, so the entries associated to them in $\widehat{M}$ are symmetric pairs of $+2$ or $+1$ entries. In case (c), if R contains the third chord, $x_1 x_{2t}$, and it is an edge in X, then we must have $end = 2t$, so R actually does not contain $x_1 x_{2t}$. Thus $x_1 x_{2t} \in R \implies x_1 x_{2t} \notin E(X)$, so the entries associated to $x_1 x_{2t}$ in $\widehat{M}$ are -1's or 0's. Case (d) is similar to case (c). □

The **switch** operation is extended to symmetric matrices as follows. Suppose $M \in \mathbb{Z}^{[k]\times[k]}$. For any $x, y \in [k]$ we define the one-edge graph $G^{x,y} = ([k]; \{xy\})$ with the adjacency matrix A_{xy}. Clearly, A_{xy} is a symmetric matrix with two 1's. Let $(x, y; z, w)$ be a list of four **pairwise distinct** elements of $[k]$. Switching along these four vertices produces the symmetric matrix

$$M' = M + A_{xz} - A_{zy} + A_{yw} - A_{wx}. \tag{6.1}$$

Clearly, the row and column sums of M' are identical to that of M. Notice, that a switch in Z translates into a switch on $\widehat{M}$.

Notice, that for bipartite degree sequences, the "top-right" submatrix of this $\widehat{M}$ is equal to the auxiliary matrix used in [28] (the bipartite adjacency matrix).

Lemma 6.2. *Let $M \in \mathbb{Z}^{[k]\times[k]}$ be a symmetric matrix with 0's in the diagonal, such that each row and column sum is in $[1, k-2]$. Also, suppose that the row sum of the first row is minimal. If the entries of M are 0 and 1, except for at most two symmetric pairs of entries of $+2$ in the first row and in the first column, and at most one symmetric pair or -1 entries anywhere in the matrix, then there exist at most* 2 *switches that transform M into a 0–1 matrix except for at most one pair of symmetric -1 entries.*

Proof. Suppose $M_{1,j} = 2$. We must have $j \neq 1$, which means that the maximum of an entry in the rest of the column of j is 1. Because the column sum is at most $k-2$ and there is at most one -1 in the column, there exist $i, i' \in [k] \setminus \{1, j\}$ such that $i \neq i'$ and $M_{i,j}, M_{i',j} \in \{-1, 0\}$. We have two cases.

(I) Suppose that there exists $\ell \in [k] \setminus \{1, i\}$ such that $M_{i,\ell} > M_{1,\ell}$. Since $1 \neq i, \ell$, we assume that $M_{i,\ell} < 2$, therefore $M_{1,\ell} \in \{0, -1\}$. Switch along $(1, i; \ell, j)$ in

M. The operation decreases $M_{1,j}$ to 1. If $M_{i,\ell} = 0$ then $M_{1,\ell} = -1$, so when the switch creates a symmetric pair of -1's, it also eliminates another pair. The matrix resulting from the switch operation satisfies the assumptions of this lemma and contains two fewer $+2$ entries.

(II) Otherwise, for all $\ell \in [k] \setminus \{1, i\}$ we have $M_{i,\ell} \le M_{1,\ell}$. Since the row sum of the first row is minimal, we have

$$\sum_{\ell=1}^{k} M_{1,\ell} \le \sum_{\ell=1}^{k} M_{i,\ell}$$

and hence

$$0 \le \sum_{\ell \in \{1,i,j\}} (M_{i,\ell} - M_{1,\ell}) = M_{i,1} - M_{1,i} + M_{i,j} - M_{1,j} = M_{i,j} - 2,$$

because M is symmetric with 0 diagonal. The inequality implies that $M_{i,j} = 2$, so either $i = 1$ or $j = 1$, which contradicts our choice of i, j.

By recursion a second pair of entries which equal $+2$ can also be eliminated. □

7. Applications of the unified method

In this Section we harvest some fruits of our unified machinery, proving a rather general result for all typical degree sequence types.

In 1990 Jerrum and Sinclair published a very influential paper [23] about fast uniform generation of regular graphs and about realizations of degree sequences where no degree exceeds $\sqrt{n/2}$. To achieve this goal, they applied the Markov chain they have developed in [22]. Informally it is known as **JS chain**, and it is sampling the perfect and near-perfect 1-factors on the corresponding Tutte gadget. The fast mixing nature of the JS chain depends on the ratio of the number of perfect and the number near-perfect 1-factors. As they proved it is applicable if and only if the degree sequence $\mathbf{d}$ belongs to a ***P*-stable** class.

Recall the definition of P-stability (introduced in Definition 1.2). Careful examination of the known results about rapidly mixing switch Markov chains revealed the fact that all known "good" degree sequence classes (for UC degree bipartite or directed degree sequences) are P-stable. It raises the conjecture that the switch Markov chains on P-stable degree classes are rapidly mixing. We resolve this conjecture affirmatively in this section.

For a fixed $B \in \mathbb{B}$, let the set of compatible auxiliary structures be

$$\mathcal{M}_B = \Big\{ \widehat{M} \ : \ \exists X, Y, Z \in \mathbb{G}(\mathbf{d}),\ s \in S_{X,Y} \\ \text{s.t. } \widehat{M} = A_X + A_Y - A_Z,\ B = B(X, Y, Z, s) \Big\}.$$

To apply the simplified Sinclair's method, it is sufficient to estimate from above the value of

$$\sum_{X,Y\in V(\mathbb{G})} \frac{\left|\left\{s \in S_{X,Y} : Z \in \Upsilon(X,Y,s)\right\}\right|}{|S_{X,Y}|} \tag{7.1}$$

for any realization Z.

Lemma 7.1. *The following bound holds for any unconstrained, bipartite, and directed degree sequence:*

$$\sum_{X,Y\in V(\mathbb{G})} \frac{\left|\left\{s \in S_{X,Y} : Z \in \Upsilon(X,Y,s)\right\}\right|}{|S_{X,Y}|} \leq n^4 \cdot \sum_{B\in\mathbb{B}} |\mathcal{M}_B|.$$

Proof. According to Lemma 4.19, expression (7.1) can be rewritten as follows:

$$\sum_{X,Y\in V(\mathbb{G})} \frac{\left|\left\{\Psi_Z(\widehat{M}(X,Y,Z), B(X,Y,Z,s), s(X,Y,Z)) : Z \in \Upsilon(X,Y,s)\right\}\right|}{|S_{X,Y}|} \tag{7.2}$$

Observe, that $|S_{X,Y}|$ is already determined by $\nabla = X\triangle Y$, which in turn is determined by Z and $\widehat{M}$. Let $t_\nabla := |S_{X,Y}|$. Furthermore, Z' is determined by B and Z (see Equations (4.11) and (4.16)). Let

$$\mathbb{B}_{\widehat{M}} = \left\{B(X,Y,Z,s) \ : \exists X,Y \text{ such that } \widehat{M} = A_X + A_Y - A_Z,\ s \in S_{X,Y}\right\}.$$

Continue writing (7.2) as follows and apply Lemma 4.16.

$$\begin{aligned} &= \sum_{\widehat{M}\in\mathcal{M}} \frac{\left|\left\{\Psi_Z(\widehat{M},B,s^*) \ : \ \exists B \in \mathbb{B},\ s^* \in \mathbb{S}(\nabla\cap Z', \nabla\setminus Z')\right\}\right|}{t_\nabla} \leq \\ &\leq \sum_{\widehat{M}\in\mathcal{M}} \frac{|\mathbb{B}_{\widehat{M}}|\cdot n^4 \cdot t_\nabla}{t_\nabla} \leq n^4 \cdot \sum_{\widehat{M}\in\mathcal{M}} |\mathbb{B}_{\widehat{M}}| \leq n^4 \cdot \sum_{B\in\mathbb{B}} |\mathcal{M}_B|. \end{aligned}$$

□

At this point, the proofs for unconstrained, bipartite, and directed degree sequences slightly diverge. The most general of these is the case of unconstrained degree sequences. First, we discuss this case. Having understood the argument, it is relatively simple to fit it to the cases of the bipartite and directed degree sequence cases. Moreover, the tools required for proving our results on the latter two classes have already been published in [28], so their proofs will be less verbose than the next section on unconstrained degree sequences.

7.1. Unconstrained degree sequences

First, let us bound the number of auxiliary structures compatible with a given parameter set.

Lemma 7.2. *If the stability of an unconstrained degree sequence* $\mathbf{d}$ *is bounded by the polynomial* $p(n)$*, then*

$$|\mathcal{M}_B| \leq n^6 \cdot p(n) \cdot |\mathbb{G}(\mathbf{d})|$$

holds for any $B \in \mathcal{B}$*.*

Proof. Equation (3.8) defines $B = (x_1, \sigma, R, w)$. Recall, that $(Z \triangle R) \triangle X \subseteq X \triangle Y$, so the bad entries ($+2$ and -1 values) in $\widehat{M}$ correspond to positions assigned to chords in R. Let M be the symmetric submatrix of $\widehat{M}$ induced by the vertices of C_r^k as rows and columns. We have two cases.

Case 1: $\boldsymbol{\nexists f \in R}$ **where** $\boldsymbol{x_1 \notin f}$ **and** $\boldsymbol{f \notin E(C_r^k)}$

All of the non 0–1 entries of $\widehat{M}$ are contained in M, in the rows and columns associated to x_1. Lemma 6.1 and Assumption (4.9) implies that we can use Lemma 6.2 to remove the $+2$'s from $\widehat{M}$ with at most two switches. For each switch, the type-(2) chord determines two vertices of the switch, thus there are n^4 ways to choose the at most two switches that eliminate the $+2$ entries.

Let $\widehat{M}'$ be the matrix we get after applying the switches defined by Lemma 6.2. Either $\widehat{M}'$ is an adjacency matrix of a realization of $\mathbf{d}$, or $\widehat{M}'$ contains -1 entries at positions associated to the chord xy. In the former case $\widehat{M}' \in \mathbb{G}(\mathbf{d})$, and in the latter $\widehat{M}' + A_{xy} + A_{yx} \in \mathbb{G}(\mathbf{d} + \mathbb{1}_x + \mathbb{1}_y)$.

Case 2: $\boldsymbol{\exists! f \in R}$ **such that** $\boldsymbol{x_1 \notin f}$ **and** $\boldsymbol{f \notin E(C_r^k)}$

This is only possible if R falls under case (e) of Lemma 2.6. The auxiliary structure belonging to the intermediate realization before or after Z on the switch sequence is one switch away from $\widehat{M}$, moreover this switch touches f. There are at most n^2 switches satisfying these conditions, because f already determines two vertices. After performing the appropriate switch on $\widehat{M}$, the enumeration of the previous case applies. □

We are ready to prove one of the main results of this paper.

Theorem 7.3. *The switch Markov chain is rapidly mixing on P-stable unconstrained degree sequence classes.*

Proof. From Lemma 7.2 and Lemma 4.18 we get:

$$n^4 \cdot \sum_{B \in \mathbb{B}} |\mathcal{M}_B| \leq n^4 \cdot |\mathbb{B}| \cdot n^6 \cdot p(n) \cdot |\mathbb{G}(\mathbf{d})| \leq \mathcal{O}(n^{18}) \cdot p(n) \cdot |\mathbb{G}(\mathbf{d})|.$$

Equation 3.9 now follows from Lemma 7.1. Every condition of the simplified Sinclair's method is satisfied, so the switch Markov chain on $\mathbb{G}(\mathbf{d})$ is rapidly mixing. □

7.2. Bipartite degree sequences

Let $\mathbf{D}$ denote a bipartite degree sequence on $n = n_1 + n_2$ vertices.

Definition 7.4. Let $\mathcal{D}$ be an infinite set of bipartite degree sequences. We say that $\mathcal{D}$ is ***P*-stable**, if there exists a polynomial $p \in \mathbb{R}[x]$ such that for any $n_1, n_2 \in \mathbb{N}, n_1 \geq n_2$, and any degree sequence $\mathbf{D} \in \mathcal{D}$ on n_1 and n_2 vertices we have

$$\left| \mathbb{G}(\mathbf{D}) \cup \left(\bigcup_{x \in [n_1],\ y \in [n_2]} \mathbb{G}(\mathbf{D} + \mathbb{1}_x + \mathbb{1}_{(n_1+y)}) \right) \right| \leq p(n) \cdot |\mathbb{G}(\mathbf{D})|,$$

where $\mathbb{1}_x$ is the x^{th} unit vector.

Recall that a primitive alternating circuit on a bipartite graph is a cycle. Also, for any $X, Y, Z \in \mathbb{G}(\mathbf{D})$, the auxiliary structure $\widehat{M} = A_X + A_Y - A_Z$ is determined by the submatrix spanned by $U \times V \subset (U \uplus V)^2$. This is the "top-right" submatrix, often called the bipartite adjacency matrix. The "top-left" and the "bottom-right" submatrices are zero.

Lemma 7.5. *If the stability of a bipartite degree sequence* $\mathbf{D}$ *is bounded by the polynomial* $p(n)$*, then*

$$|\mathcal{M}_B| \leq n^4 \cdot p(n) \cdot |\mathbb{G}(\mathbf{D})|$$

holds for any $B \in \mathcal{B}$*.*

Proof. The proof is simpler and slightly different than that of Lemma 7.2. Double step is never called in the bipartite case (Lemma 2.5), so either R is empty or Lemma 6.1 applies to it. Hence, there are $(n_1 n_2)^2$ possibilities to choose the remaining vertices of the (at most two) switches that eliminate the type-(2) bad chords.

Secondly, we have to make sure that the switches produced by Lemma 6.2 respect the bipartition. As before, let M be the the submatrix of $\widehat{M}$ induced by the vertices of C_r^k. Let $H = K_{U(C_r^k)} \uplus K_{V(C_r^k)}$ be the disjoint union of the two cliques within the classes. Instead of applying Lemma 6.2 on M, apply it on $M + A_H$. Each row and column sum increased by the same number, therefore assumptions of the lemma are still satisfied. Any switch which eliminates a +2 from this matrix which is valid in the unconstrained sense also respects the bipartition. □

Theorem 7.6. *The switch Markov chain is rapidly mixing on* P*-stable bipartite degree sequence classes.*

Proof. Instead of Lemma 7.2 we use Lemma 7.5. The bound on the size of $\mathbb{B}$ is $\mathcal{O}(n^6)$ according to Lemma 4.19. Other than these differences, the proof is identical to that of Theorem 7.3:

$$n^4 \cdot \sum_{B \in \mathbb{B}} |\mathcal{M}_B| \leq n^4 \cdot |\mathbb{B}| \cdot n^4 \cdot p(n) \cdot |\mathbb{G}(\mathbf{D})| \leq \mathcal{O}(n^{14}) \cdot p(n) \cdot |\mathbb{G}(\mathbf{D})|.$$

□

7.3. Directed degree sequences

Recall from Section 5 that instead of directly manipulating directed graphs, we work on their bipartite representations. Formally, the degree sequence of the directed graph is identical to that of its bipartite representation. Through the bipartite representation, directed degree sequence classes inherit a definition of P-stability (thus the realizations of the representing [perturbed] bipartite degree sequences also avoid the non-chords). Let $\vec{\mathbf{D}}$ denote the bipartite representation of a directed degree sequence $\vec{\mathbf{d}}$ on n vertices (so the length of the vector $\vec{\mathbf{D}}$ is $2n$).

Lemma 7.7. *If the stability of a directed degree sequence $\vec{\mathbf{d}}$ is bounded by the polynomial $p(n)$, then*

$$|\mathcal{M}_B| \leq n^4 \cdot p(n) \cdot |\mathbb{G}(\vec{\mathbf{d}})| = n^4 \cdot p(n) \cdot |\mathbb{G}(\vec{\mathbf{D}})|$$

holds for any $B \in \mathcal{B}$.

Proof. The proof of Lemma 7.5 applies to the bipartite representation, but we have to check that applying Lemma 6.2 on $M + A_H$ produces switches that avoid the non-chords. Indeed, this is the case, because the non-chords of the form $u_x v_x$ correspond to the main diagonal in M, which the switches chosen by the lemma avoid. $\square$

Theorem 7.8. *The switch Markov chain is rapidly mixing on P-stable directed degree sequence classes.*

Proof. By Lemma 4.19, the bound on the size of $\mathbb{B}$ is $\mathcal{O}(n^6)$, as in the proof of Theorem 7.6. From the previous lemma, we get:

$$n^4 \cdot \sum_{B \in \mathbb{B}} |\mathcal{M}_B| \leq n^4 \cdot |\mathbb{B}| \cdot n^4 \cdot p(n) \cdot |\mathbb{G}(\vec{\mathbf{D}})| \leq \mathcal{O}(n^{14}) \cdot p(n) \cdot |\mathbb{G}(\vec{\mathbf{D}})|.$$

$\square$

8. *P*-stable degree sequence classes

In the proof of almost every previous result on rapid mixing of the switch Markov chain, it turns out there is a short hidden proof that the degree sequences under study are P-stable. The unified proof contains most of the technical difficulty of proving rapid mixing of the switch Markov chain.

There have already been successful attempts at unifying some of the proofs, most notably [1], which studies the notion of strong stability:

Definition 8.1 (adapted from [1])**.** Let $\mathcal{D}$ be a set of degree sequences. Let

$$\mathbb{G}'(\mathbf{d}) = \bigcup_{x,y \in [n]} \mathbb{G}(\mathbf{d} - \mathbb{1}_x - \mathbb{1}_y).$$

We say that $\mathcal{D}$ is strongly stable if there exists a constant ℓ such that for any $\mathbf{d} \in \mathcal{D}$ and any $G' \in \mathbb{G}'(\mathbf{d})$ there exists $G \in \mathbb{G}(\mathbf{d})$ (which depends on G') such that $|E(G') \Delta E(G)| \leq 2\ell$.

For bipartite graphs, the definition is analogous. The above definition is easily seen to be equivalent with the one given in [1], and it has the advantage that it does not rely on the definition of the Jerrum-Sinclair chain.

Theorem 8.2 (Amanatidis and Kleer [1])**.** *The switch Markov chain is rapidly mixing on strongly stable unconstrained and bipartite degree sequence classes.*

In the following subsections of this section we discuss all known P-stable degree sequence regions. It is an intriguing problem to discover other P-stable regions.

8.1. Unconstrained degree sequences

For the sake of having more readable and compact formulas, let $\Delta = \max \mathbf{d}$, $\delta = \min \mathbf{d}$, and $m = \frac{1}{2}\sum_{v\in V} \mathbf{d}(v)$ be functions of $\mathbf{d}$.

Recently, Greenhill and Sfragara [18] published a breakthrough result on the rapid mixing of the switch Markov chain.

Theorem 8.3 ([18])**.** *The switch Markov chain is rapidly mixing on the following family of unconstrained degree sequences:*

$$\mathcal{D}_{GS} := \left\{ \mathbf{d} \in \mathbb{Z}^+ \ : \ \delta \geq 1,\ 3 \leq \max \mathbf{d} \leq \frac{1}{3}\sqrt{2m} \right\} \tag{8.1}$$

It turns out that the authors implicitly prove on page 10 of [18] that $\mathcal{D}_{\mathrm{GS}}$ is a P-stable class. However, this implicit result is actually not new: Jerrum, McKay, and Sinclair extensively studied the notion of P-stability in their seminal work [24].

Theorem 8.4 (Jerrum, McKay, Sinclair — Theorem 8.1 in [24])**.** *The family of unconstrained degree sequences*

$$\mathcal{D}_{JMS} := \left\{ \mathbf{d} \in \mathbb{N}^n \ : \ (\max \mathbf{d} - \min \mathbf{d} + 1)^2 \leq 4 \cdot \min \mathbf{d} \cdot (n - \max \mathbf{d} - 1) \right\}$$

is P-stable.

Theorem 8.5 (Jerrum, McKay, Sinclair — Theorem 8.3 in [24])**.** *The family of unconstrained degree sequences*

$$\begin{aligned}\mathcal{D}_{JMS+} := \Big\{ \mathbf{d} \in \mathbb{N}^n \ &: (2m - n\delta)(n\Delta - 2m) \leq \\ &\leq (\Delta - \delta)\big((2m - n\delta)(n - \Delta - 1) + (n\Delta - 2m)\delta\big) \Big\},\end{aligned}$$

is P-stable.

Theorem 7.3 implies that the switch Markov chain is rapidly mixing on elements of $\mathcal{D}_{\mathrm{JMS}}$ and $\mathcal{D}_{\mathrm{JMS+}}$. Moreover, it is easy to see that $\mathcal{D}_{\mathrm{GS}} \subset \mathcal{D}_{\mathrm{JMS+}}$. However, the proofs of Theorems 8.4 and 8.5 actually prove a bit more than just P-stability. In [24] it is also shown that $\mathcal{D}_{\mathrm{JMS}}$ and $\mathcal{D}_{\mathrm{JMS+}}$ are **strongly stable** regions with $\ell \leq 10$, so Theorem 8.2 already applies to them.

The following corollary is a consequence of the fact that the degrees in an Erdős-Rényi random graph are tightly concentrated around their expected value.

Corollary 8.6. *Let $G(n,p)$ be an Erdős-Rényi random graph of order $n \geq 100$ with edge probability p, where p is bounded away from 0 and 1 by at least $\frac{5\log n}{n-1}$. Then* $\Pr(\mathbf{d}(G(n,p)) \in \mathcal{D}_{JMS+}) \geq 1 - \frac{3}{n}$.

Proof. We may suppose that $p \leq \frac{1}{2}$ by taking the complement of G if necessary. Let $p = p(n)$, $\varepsilon_1 = \sqrt{\frac{\log n}{n-1}}$ and $m = \frac{1}{2}\sum_{v \in V} \mathbf{d}(v) = \binom{n}{2}(p + \varepsilon_2)$. By Hoeffding's inequality, we have

$$\begin{aligned}
&\Pr\big(\Delta(G) > (p+\varepsilon_1)\cdot(n-1)\big) \leq \\
&\quad\leq \sum_{v \in V(G)} \Pr\Big(\mathbf{d}(v) < (p+\varepsilon_1)\cdot(n-1)\Big) \leq n \cdot e^{-2\varepsilon_1^2(n-1)} \leq \frac{1}{n},
\end{aligned}$$

and similarly

$$\Pr\big(\delta(G) < (p - \min(\varepsilon_1, p))\cdot(n-1)\big) \leq \frac{1}{n}.$$

The degree sequence $\mathbf{d}(G(n,p))$ is in $\mathcal{D}_{\mathrm{JMS+}}$ if it satisfies

$$(2m - n\delta)(n\Delta - 2m) \leq (\Delta - \delta)\left((2m - n\delta)(n - \Delta - 1) + (n\Delta - 2m)\delta\right). \tag{8.2}$$

First suppose that $p \geq \varepsilon_1$. Because increasing Δ or decreasing δ makes the inequality stricter, without loss of generality, we may substitute $\Delta = (p+\varepsilon_1)\cdot(n-1)$ and $\delta = (p-\varepsilon_1)\cdot(n-1)$ into (8.2). We calculate

$$\begin{aligned}
(2m - n\delta) &= \left(2\binom{n}{2}(p+\varepsilon_2) - n(p+\varepsilon_1)(n-1)\right) = n(n-1)(\varepsilon_1+\varepsilon_2),\\
(n\Delta - 2m) &= n(n-1)(\varepsilon_1+\varepsilon_2),\\
(2m - n\delta)(n - \Delta - 1) &= n(n-1)(\varepsilon_1+\varepsilon_2)\cdot(1 - p - \varepsilon_1)(n-1),\\
(n\Delta - 2m)\delta &= n(n-1)(\varepsilon_1+\varepsilon_2)\cdot(p - \varepsilon_1)(n-1).
\end{aligned}$$

Therefore (8.2) holds if

$$\left(n(n-1)(\varepsilon_1+\varepsilon_2)\right)^2 \leq 2\varepsilon_1(n-1)\cdot n(n-1)(\varepsilon_1+\varepsilon_2)\cdot(1-2\varepsilon_1)(n-1),$$

or after simplification,

$$n\varepsilon_2 \leq (n-2)\varepsilon_1 - 4\varepsilon_1^2(n-1).$$

If $\varepsilon_2 \leq \frac{\sqrt{\log n}}{n-1}$ then the last inequality is satisfied whenever

$$n\frac{\sqrt{\log n}}{n-1} \leq (n-2)\sqrt{\frac{\log n}{n-1}} - 4\log n.$$

Clearly, the right hand side grows $\Theta(n^{\frac{1}{2}})$ faster than the left hand side as $n \to \infty$, and the inequality already holds for $n = 100$. Now for any $c > 0$,

$$\Pr\left(\frac{1}{2}\sum_{v \in V} d(v) > (p+c)\binom{n}{2}\right) \leq e^{-2c^2\binom{n}{2}},$$

and substituting $c = \frac{\sqrt{\log n}}{n-1}$ shows that $\varepsilon_2 \leq \frac{\sqrt{\log n}}{n-1}$ with probability at least $1 - 1/n$. Overall, $\Pr(\mathbf{d}(G(n,p)) \notin \mathcal{D}_{\mathrm{JMS+}}) \leq \frac{1}{n} + \frac{1}{n} + \frac{1}{n}$ if $p \geq \varepsilon_1$.

Now suppose that $\frac{5 \log n}{n-1} \leq p < \varepsilon_1$. Substituting $\Delta = (p + \varepsilon_1) \cdot (n-1)$ and $\delta = 0$ into (8.2), we find that $\mathbf{d}(G(n,p)) \in \mathcal{D}_{\mathrm{JMS+}}$ if

$$2m(n\Delta - 2m) \leq \Delta \cdot 2m(n - \Delta - 1).$$

Rearranging, this inequality holds if $\Delta(\Delta + 1) \leq 2m$, and substituting for Δ and simplifying gives the sufficient condition $4\varepsilon_1^2 \leq p - \varepsilon_2$.

The last inequality is satisfied if $\varepsilon_2 = \frac{\sqrt{\log n}}{n}$. Therefore, if $\frac{5 \log n}{n-1} \leq p < \varepsilon_1$, then $\Pr(\mathbf{d}(G(n,p)) \notin \mathcal{D}_{\mathrm{JMS+}}) \leq \frac{2}{n}$. $\square$

Similar results have already been proved for bipartite Erdős-Rényi graphs [10, 11], with the requirement that p is bounded away from 0 and 1 by at least $4\sqrt{\frac{2 \log n}{n}}$.

8.2. Unconstrained power-law bounded degree sequences

The switch Markov chain is not the only way to exactly sample the uniform distribution on the realizations of a degree sequence. Recently, Gao and Wormald presented the first "provably practical" sampler for power-law distribution-bounded degree sequence where γ is allowed to be less than 3; in fact they can go as low as 2.8811. For such degree sequences, they provide a *linear time* approximate sampler and a polynomial time exact sampler.

In degree distributions of empirical networks following a power-law, the parameter γ is usually between 2 and 3.

Gao and Wormald also enumerate the number of realizations of several heavy-tailed degree sequences [14]. In particular, they calculate the number of realizations of degree sequences that are

(1) power-law **density-bounded** with $\gamma > 2.5$,

(2) power-law **distribution-bounded** with $\gamma > 1 + \sqrt{3} \approx 2.732$.

Their analysis of degree sequences obeying (1) shows that they are contained by $\mathcal{D}_{\mathrm{GS}}$. The formula in [14] enumerating the realizations of degree sequences obeying (2) directly implies P-stability, thus our Theorem 7.3 applies. Therefore Theorem 7.3 provides a fully polynomial time almost uniform sampler:

Theorem 8.7 (Follows from Theorem 7.3 and [15])**.** *The switch Markov chain is rapidly mixing on unconstrained degree sequences satisfying a power-law distribution-bound for any* $\gamma > 1 + \sqrt{3}$.

Gao and Wormald [14] compute the number of realizations for other types of heavy-tailed degree sequences, and in-turn, those formulas imply P-stability of the respective classes. They also conjecture that degree sequences obeying a power-law distribution-bound with $\gamma > 2$ are P-stable [15].

8.3. Bipartite degree sequences

Let $\mathbf{D}$ be a bipartite degree sequence on U and V as color classes. We use the following shorthands in this sub-section:

$$\delta_U = \min_{u\in U} \mathbf{D}(u), \qquad \delta_V = \min_{v\in V} \mathbf{D}(v),$$
$$\Delta_U = \max_{u\in U} \mathbf{D}(u), \qquad \Delta_V = \min_{v\in V} \mathbf{D}(v),$$

and $m = \sum_{u\in U} \mathbf{D}(u) = \sum_{v\in v} \mathbf{D}(v)$.

Theorem 8.8 (implicitly proved in Theorem 2 in [11])**.** *The set of bipartite degree sequences $\mathbf{D}$ that satisfy*

$$2 \le \Delta \le \sqrt{\frac{m}{2}}, \tag{8.3}$$

is P-stable.

Clearly, Theorems 8.3 and 8.8 are closely related, the difference in constants is caused by the different structural constraints only.

Theorem 8.9 (implicitly proved in Theorem 3 in [11])**.** *The set of bipartite degree sequences $\mathbf{D}$ that satisfy*

$$(\Delta_U - \delta_U - 1)(\Delta_V - \delta_V - 1) \le \max\Big(\delta_U(|U| - \Delta_V + 1), \delta_V(|V| - \Delta_U + 1)\Big) \tag{8.4}$$

is P-stable.

Amanatidis and Kleer recently presented a bipartite analogue of Theorem 8.4.

Theorem 8.10 (Corollary 18 in [1])**.** *The set of bipartite degree sequences that satisfy both*

$$\begin{aligned}(\Delta_U - \delta_V)^2 &\le 4\delta_V \cdot (|V| - \Delta_U)\\ (\Delta_V - \delta_U)^2 &\le 4\delta_U \cdot (|U| - \Delta_V)\end{aligned} \tag{8.5}$$

is P-stable (because it is strongly stable).

The following Theorem 8.11 is a bipartite analogue of Theorem 8.4. In some sense, it is stronger than either Theorem 8.9 or 8.10. If one side is regular ($\Delta_U = \delta_U$), then Inequality (8.4) and (8.6) are automatically satisfied. Inequality (8.4) trivially holds even for almost half-regular bipartite degree sequences ($\Delta_U \le \delta_U + 1$). Assuming $|U| = |V|$, $\Delta_U = \Delta_V$, $\delta_U = \delta_V$ are all satisfied, (8.5) and (8.6) are equivalent, and are loosely speaking 4 times better than (8.4).

Theorem 8.11. *The set of bipartite degree sequences that satisfy*

$$(\Delta_U - \delta_U) \cdot (\Delta_V - \delta_V) \le 4 \cdot \min\Big(\delta_U(|U| - \Delta_V), \delta_V(|V| - \Delta_U)\Big) \tag{8.6}$$

is P-stable (because it is strongly stable).

Proof. If the degrees of two vertices in the same class are increased by one in a graphic bipartite degree sequence, then the resulting degree sequence is not graphic. Let G be a realization of $\mathbf{D}+\mathbb{1}_{u_1}+\mathbb{1}_{v_1}$ on U and V as vertex classes. The degree sequence of G is

$$\mathbf{D}(G)=\begin{cases} d(x)+1 & \text{if } x=u_1 \text{ or } v_1,\\ d(x) & \text{otherwise.}\end{cases}$$

We claim that there exists an alternating path P of length at most 7 between u_1 and v_1 in G, such that the first and last edges of P are edges of G. Assume that no such path exists. Let U_1, V_2, U_3 be the set of vertices that are reachable from v_1 via an alternating path (starting with and edge of v_1 in G) of length exactly 1, 2, and 3, respectively. Define V_1, U_2, V_3 similarly with respect to u_1.

By our assumption, $u_1 \notin U_1$, $v_1 \notin V_1$, otherwise $\{u_1, v_1\} \in E(G)$ is an alternating path of length 1. If $G[U_1, V_1]$ is not a complete bipartite graph, then there is an alternating path of length 3 which is a good candidate for P. Similarly, $G[U_2, V_2]$ is an empty graph (no alternating paths of length 5), and $G[U_3, V_3]$ is a complete bipartite graph (no alternating paths of length 7). These observation also imply that $\{u_1\} \uplus U_1 \uplus U_2 \uplus U_3$ and $\{v_1\} \uplus V_1 \uplus V_2 \uplus V_3$ are subpartitions of U and V, respectively. Let U_4 and V_4 be the remaining vertices of U and V, respectively.

By definition and the fact that $G[U_3, V_3]$ is complete bipartite graph, $G[U_1 \cup U_3, V_1 \cup V_3]$ is also a complete bipartite graph. The vertices in U_2 are only adjacent to elements of $V_1 \cup V_3$, therefore

$$|E(U_2, V_1 \cup V_3)| = |E(U_2, V)| \geq \delta_U |U_2|.$$

Every vertex which is joined by a non-edge to a vertex of V_1 is contained in U_2. Therefore $G[U_4, V_1]$ is a complete bipartite graph and $|V_1| = d(u_1) \geq \delta_U + 1$, thus

$$|E(U_4, V_1 \cup V_3)| \geq |E(U_4, V_1)| > \delta_U |U_4|.$$

Since $G[U_1 \cup U_3, V_1 \cup V_3]$ is a complete bipartite graph, we have

$$|E(U_2 \cup U_4, V_1 \cup V_3)| \leq |V_1 \cup V_3| \cdot (\Delta_V - |U_1 \cup U_3|)\,.$$

Combining the previous inequalities, we get

$$\delta_U(|U| - |U_1 \cup U_3| - 1) = \delta_U \cdot |U_2 \cup U_4| < |V_1 \cup V_3| \cdot (\Delta_V - |U_1 \cup U_3|)\,.$$

Let us substitute $k_1 = |U_1 \cup U_3|$ and $k_2 = |V_1 \cup V_3|$, leading to

$$\begin{aligned} \delta_U(|U| - \Delta_V - 1) &< (\Delta_V - k_1) \cdot (k_2 - \delta_U),\\ \delta_V(|V| - \Delta_U - 1) &< (\Delta_U - k_2) \cdot (k_1 - \delta_V). \end{aligned}$$

The second inequality is obtained by symmetry. Solving for k_1, we get

$$\frac{\delta_V(|V|-\Delta_U-1)}{\Delta_U-k_2}+\delta_V<k_1<\Delta_V-\frac{\delta_U(|U|-\Delta_V-1)}{k_2-\delta_U}.$$

Without loss of generality, we may assume that $\delta_V(|V|-\Delta_U-1)\leq\delta_U(|U|-\Delta_V-1)$. Omitting k_1 from the middle of the above inequality, we have

$$\delta_V(|V|-\Delta_U-1)(\Delta_U-\delta_U)<(\Delta_V-\delta_V)(\Delta_U-k_2)(k_2-\delta_U).$$

The left hand side in the last inequality is maximal if $k_2=\frac{1}{2}(\Delta_U+\delta_U)$. We get

$$4\delta_V(|V|-\Delta_U-1)<(\Delta_V-\delta_V)(\Delta_U-\delta_U),$$

which contradicts the assumptions of this theorem. Thus there exists a suitable alternating path of length at most 7 starting on an edge of u_1 and ending on an edge of v_1.

Switching the edges along the alternating path P transforms G into a realization of $\mathbf{D}$. The procedure consists of at most 8 non-deterministic choices (vertices of the alternating path), so $p(|U|+|V|)=(|U|+|V|)^8$ is a good witness to stability. □

8.4. Directed degree sequences

Let $\vec{\mathbf{d}}$ be a directed degree sequence on X as vertices. Let $\vec{\mathbf{d}}_{\text{out}}$ be the out-degree sequence and $\vec{\mathbf{d}}_{\text{in}}$ be the in-degree sequence. We use the following abbreviations in this sub-section:

$$\delta_{\text{out}}=\min_{x\in X}\vec{\mathbf{d}}_{\text{out}}(x),\qquad \delta_{\text{in}}=\min_{x\in X}\vec{\mathbf{d}}_{\text{in}}(x),$$

$$\Delta_{\text{out}}=\min_{x\in X}\vec{\mathbf{d}}_{\text{out}}(x),\qquad \Delta_{\text{in}}=\min_{x\in X}\vec{\mathbf{d}}_{\text{in}}(x),$$

and $m=\sum_{x\in X}\vec{\mathbf{d}}_{\text{out}}(x)=\sum_{x\in X}\vec{\mathbf{d}}_{\text{in}}(x)$.

Theorem 8.12 (implicitly proved in [18])**.** *The set of bipartite degree sequences* $\vec{\mathbf{d}}$ *that satisfy*

$$2\leq\max(\Delta_{\text{out}},\Delta_{\text{in}})\leq\frac{1}{4}\sqrt{m},\tag{8.7}$$

is P-stable.

Theorem 8.13 (implicitly proved in Theorem 4 in [11])**.** *The set of directed degree sequences* $\vec{\mathbf{d}}$ *satisfying*

$$2\leq\max(\Delta_{\text{out}},\Delta_{\text{in}})<\frac{1}{\sqrt{2}}\sqrt{m-4},$$

is P-stable.

Theorem 8.14 (implicitly proved in Theorem 5 in [11])**.** *The set of directed degree sequences* $\vec{\mathbf{d}}$ *satisfying*

$$(\Delta_{\text{out}} - \delta_{\text{out}}) \cdot (\Delta_{\text{in}} - \delta_{\text{in}}) \le 2 - n +$$
$$+ \max\Big(\delta_{\text{out}}(n - \Delta_{\text{in}} - 1) + \delta_{\text{in}} + \Delta_{\text{out}}, \delta_{\text{in}}(n - \Delta_{\text{out}} - 1) + \delta_{\text{out}} + \Delta_{\text{in}}\Big)$$

is P-stable.

9. Summary

To summarize the new results of the paper we present Table 2, an updated version of Table 1 which contains both entirely new and improved results. A strongly stable class is, as the name suggests, naturally P-stable, see the recent paper of Amanatidis and Kleer [1]. Their results already provide a unified framework for proving all previously known bipartite and UC degree sequence results.

UC degree sequences	bipartite deg. seq.	directed deg. seq.
regular [3]	(half-)regular [28]	regular [16]
	almost half regular [8]	
$\Delta \le \frac{1}{3}\sqrt{2m}$ [18]	$\Delta \le \frac{1}{\sqrt{2}}\sqrt{m}$ [11]	$\Delta < \frac{1}{\sqrt{2}}\sqrt{m-4}$ [11]
Power-law distribution-bound, $\gamma > 1 + \sqrt{3}$		
$(\Delta - \delta + 1)^2 \le$ $\le 4 \cdot \delta(n - \Delta - 1)$ proof in [1]	$(\Delta_U - \delta_U) \cdot (\Delta_V - \delta_V) \le$ $4\delta_U(\lvert U\rvert - \Delta_V), 4\delta_V(\lvert V\rvert - \Delta_U)$ Theorems 8.9 and 8.10	similar to bipartite case [10, 11]
Erdős-Rényi $G(n,p)$ $p, 1-p \ge \frac{5 \log n}{n-1}$	Bipartite Erdős-Rényi [10, 11] $p, 1-p \ge 4\sqrt{\frac{2 \log n}{n}}$	similar to bipartite case [10, 11]
strongly stable degree sequence classes [1]		
***P*-stable degree sequence classes**		

Table 2: Updated version of Table 1 with the new results in this paper. Here Δ and δ denote the maximum and minimum degrees, respectively. Half of the sum of the degrees is m, and n is the number of vertices. The notation is similar for bipartite and directed degree sequences. Some technical conditions have been omitted. Gray text is used for previously known results.

The flexibility of our unified method allowed us to extend the rapid mixing results of the switch Markov chain in two directions (in the table):

- vertically (power of machinery) to P-stable degree sequence classes, and
- horizontally (applicability of machinery) to directed degree sequences.

Theorem 8.7 extends the class of power-law like degree sequences having an almost uniform sampler to include those sequences which follow a power-law distribution-bound with $\gamma > 1+\sqrt{3}$. Gao and Wormald [15] conjecture that power-law distribution-bounded degree sequences with $\gamma > 2$ are P-stable. Empirical evidence suggests that this latter class contains most real-world networks following a power-law [14].

We have also shown that the degree sequence of the Erdős-Rényi random graph $G(n, p)$ is rapidly mixing with high probability as $n \to \infty$, for any edge probability p satisfying $p, 1-p \geq \frac{5\log n}{n-1}$.

The notion of P-stability arises naturally when studying the rapid mixing of the switch Markov chain [23, 24]. It would be really intriguing to find even a small rapidly mixing degree sequence class which is not P-stable. Finding the bipartite and directed analogues of Theorem 8.5 seems to be a relatively easy and moderately rewarding open problem.

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