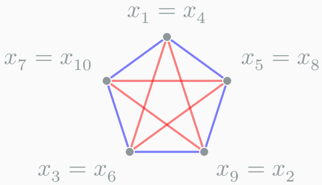
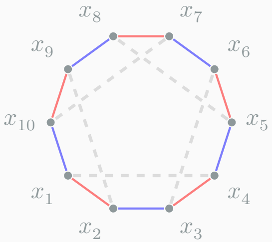
On the mixing time of switch Markov chains





Tamás Róbert Mezei

Combinatorics seminar, 29 October 2020

Alfréd Rényi Insitute of Mathematics

Introduction, background

Problem statement

- Given a non-negative integer sequence d of even sum, $\mathcal{G}(d)$ be the set of simple graph with degree sequence d (vertices are labeled)

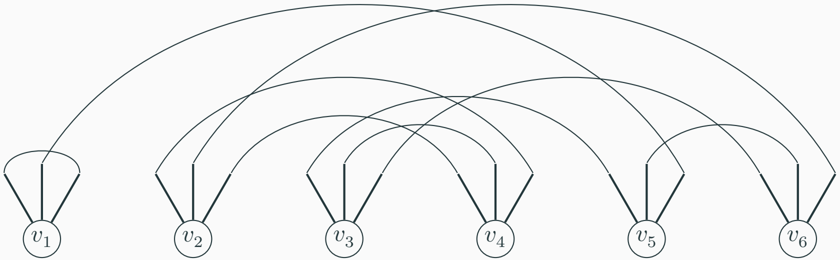
- *Problem*: take a sample $G \in \mathcal{G}(d)$ uniformly and randomly $(\min d \geq 1, \max d \leq n-1)$

- *Motivation*:
  - Randomized approximate counting: Jerrum, Valiant, Vazirani (1986)
  - Hypothesis testing, statistics
    - There is usually only one observed network, so experiments cannot be repeated
    - Null model: structure of network explained by the properties of the deg. sequence
    - Via sampling, statistical parameters of the null model can be measured
  - Benchmarking software, algorithms, simulations (network science)

Stub pairing - configuration model

- Given a degree sequence $d = (d_1, d_2, \dots, d_n)$, take v_i with d_i half-edges $\forall i \in [1, n]$

Stub pairing - configuration model



- Given a degree sequence $d = (d_1, d_2, \ldots, d_n)$, take v_i with d_i half-edges $\forall i \in [1, n]$
- Take a random complete matching between the half-edges ($\|d\|_1 \in 2\mathbb{N}$).

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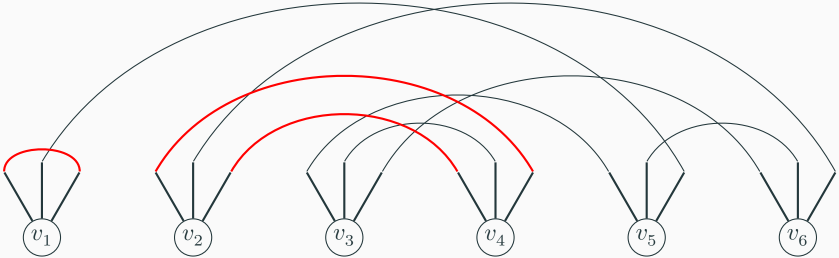

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- Take a random complete matching between the half-edges ($\|d\|_1 \in 2\mathbb{N}$).
- The resulting object may contain loops and multiedges.

Stub pairing - configuration model

Theorem (Bollobás, 1980)

The number of r-regular simple graphs on n (labeled) vertices is asymptotic to

$$e^{-\lambda-\lambda^2} \cdot \frac{(2m)!}{m!2^m(r!)^n},$$

where $\lambda = \frac{1}{2}(r-1)$ and $m = \frac{1}{2}rn$.
(Bollobás actually enumerated most simple graphs with $\Delta \leq \sqrt{2\log n} - 1$).

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- A. Békéssy, P. Békéssy, J. Komlós 1972: asymptotic enumeration of p, q-regular $m+n$ vertex bipartite graphs

Stub pairing - rejection schemes

The algorithms below start with a sample from the configuration model then try to fix loops and multiedges; if they can't, the sample is **rejected**.

Theorem (McKay and Wormald, 1990)

Uniformly generate simple graphs satisfying $\Delta \leq \mathcal{O}\left(m^{\frac{1}{4}}\right)$ *in* $\mathcal{O}(\Delta^4 n^2)$ *expected time.*
Uniformly generate r*-regular graphs for* $r = o\left(n^{\frac{1}{3}}\right)$ *in* $\mathcal{O}(nr^3)$ *expected time.*

Theorem (Gao and Wormald, 2017)

Uniformly generate r*-regular graphs for* $r = o(\sqrt{n})$ *in* $\mathcal{O}(nr^3)$ *expected time.*

Theorem (Gao and Wormald, 2018)

Uniformly generate graphs whose degree sequence obeys a power-law distribution bound for some $\gamma > 2.8811$.

Exact vs. approximate sampling

- The previous results all depend on the asymptotic counting of the number of realizations of the respective degree sequences

- Instead of exactly sampling the uniform distribution on $\mathcal{G}(d)$, allow an ε difference in total variation.

Definition (Polynomial-time approximate uniform sampler)

An algorithm running in $\mathrm{poly}(n) \cdot \log \varepsilon^{-1}$ (expected) time s.t. the ℓ_1-distance of the sample distribution is ε close to the uniform distribution is called a polynomial-time approximate uniform sampler.

Markov chain Monte Carlo methods

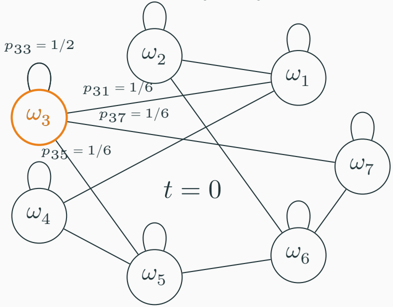
Markov Chains - a reminder

Definition (discrete time finite Markov chain)

$\mathcal{M} = (\Omega, P)$, where Ω is a finite state space, and $P = (p_{ij})_{\omega_i, \omega_j \in \Omega}$ is the transition matrix, where p_{ij} is the probability of moving from state ω_i to ω_j. Every step taken by the chain is independent from its previous steps and $\sum_j p_{ij} = 1 \; \forall i$.

- π_0: initial prob. distribution on Ω
- $\pi_t = \pi_0 P^t$
- $p_{ij} = p_{ji} \forall i, j \implies \pi \equiv |\Omega|^{-1}$ is a stationary distribution: $\pi = P\pi$

$$G(\mathcal{M}) = (\Omega, \{\omega_i \omega_j \mid p_{ij} \neq 0\})$$



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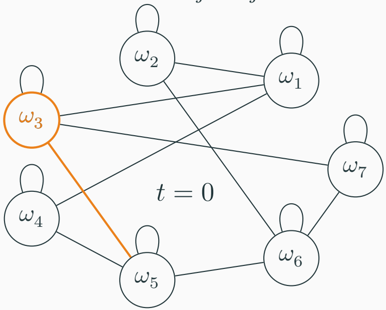

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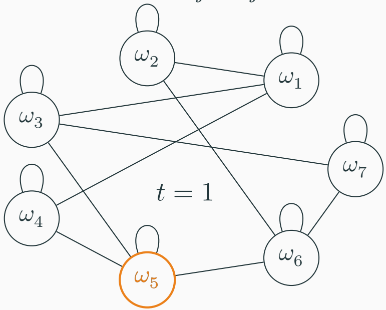

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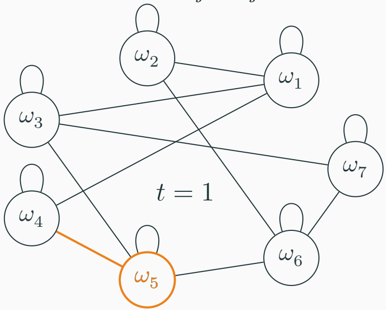

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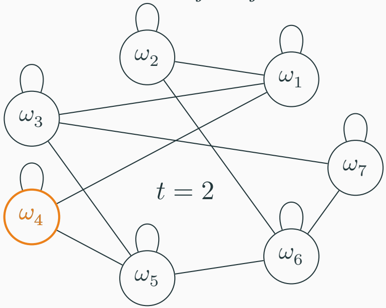

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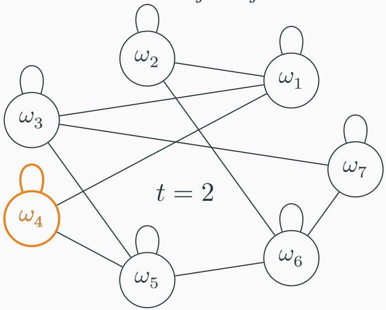

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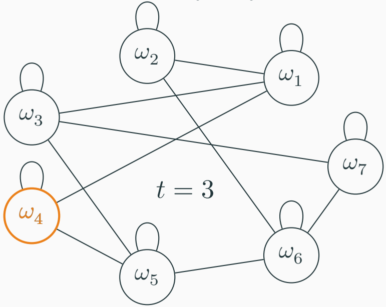

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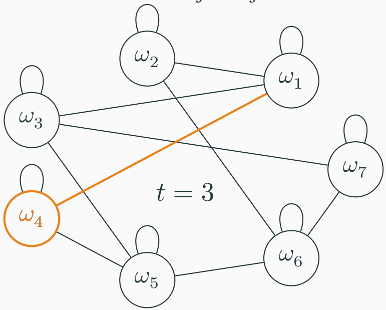

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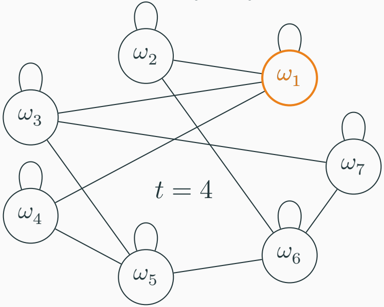


Convergence theorem, rate of convergence

Theorem

If $G(\mathcal{M})$ is connected and aperiodic, then $\pi_t(\omega_j) \to \pi(\omega_j)$ as $t \to \infty$ for any π_0.

- Convergence is exponentially quick, i.e., $|\pi_t(\omega_j) - \pi(\omega_j)| \leq \mu^t$ for some $\mu \in [0, 1)$.
- Let λ_2 be the second largest eigenvalue of P, then $\mu \leq \lambda_2$ (lazy chain)
- To get an approximate sampler, it is sufficient to have

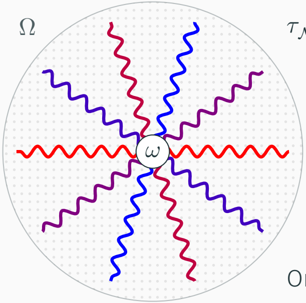
$$t \geq \frac{1}{1 - \lambda_2} (\log |\Omega| - \log \varepsilon)$$

$$t \cdot \log \lambda_2 \leq t(\lambda_2 - 1) \leq \log \varepsilon - \log |\Omega|$$

$$\boxed{|\pi_t(\omega_j) - \pi(\omega_j)| \leq (\lambda_2)^t \leq \varepsilon / |\Omega|}$$

The Sinclair method for estimating the eigenvalue-gap ($\pi \equiv 1/|\Omega|$)

Let f be a multicommodity-flow which sends a commodity of quantity 1 between each pair of states $\omega_i, \omega_j \in \Omega$ in the Markov-graph $G(\mathcal{M})$.



$\tau_{\mathcal{M}}(\varepsilon)$ is the min. time s.t. $|\pi_0 P^t - \pi| \leq \varepsilon$ holds $\forall t \geq \tau_{\mathcal{M}}(\varepsilon)$

$\rho(f)$: max amount of flow through any $\omega \in \Omega$

$\ell(f)$: max length of a flow in f

On **average**, the flow through an $\omega \in \Omega$ is at most $\ell(f)\frac{\binom{|\Omega|}{2}}{|\Omega|}$

Theorem (Sinclair, 1988)

The mixing time $\tau_{\mathcal{M}}(\varepsilon) \leq \left(\max_{p_{ij} \neq 0} \frac{1}{p_{ij}}\right) \cdot \frac{\rho(f) \cdot \ell(f)}{|\Omega|} \cdot (\log|\Omega| - \log\varepsilon)$

Rapid mixing

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The mixing time $\tau_{\mathcal{M}}(\varepsilon) \leq \left(\max_{p_{ij}\neq 0} \frac{1}{p_{ij}}\right) \cdot \frac{\rho(f)\cdot \ell(f)}{|\Omega|} \cdot (\log|\Omega| - \log\varepsilon)$

Definition (Rapid/fast mixing)

We say that a Markov-chain $\mathcal{M}$ is rapidly mixing if

$$\tau_{\mathcal{M}}(\varepsilon) \leq \mathrm{poly}(\log|\Omega|, -\log\varepsilon).$$

In our applications we will have

$$\log|\Omega| \leq \log(n^{\alpha}|\mathcal{G}(d)|) \leq \alpha\log n + m\log 2m,$$

thus rapid mixing implies that there exists a polynomial time approximate sampler.

Jerrum-Sinclair chain

State space: $\Omega = \mathcal{G}(d) \cup \bigcup_{i,j \in V} \mathcal{G}(d + \mathbb{1}_i + \mathbb{1}_j)$.

Transitions: u.a.r. choose $a, b \in V(G)$, then

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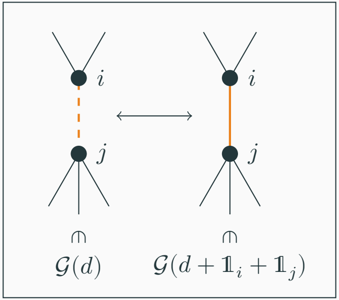

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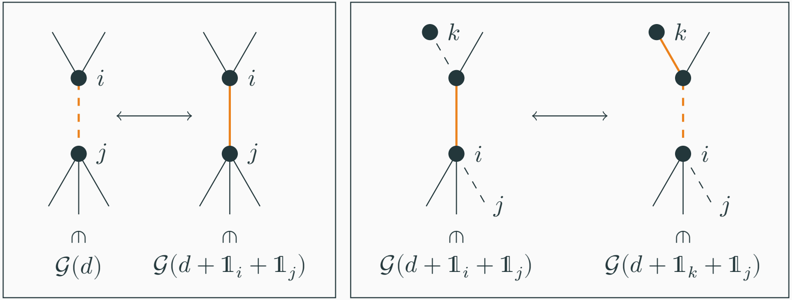
- if $G \in \mathcal{G}(d)$ and $ab \notin E(G)$, add ab to G.
- if $G \notin \mathcal{G}(d)$, $ab \in E(G)$, and $\deg_G(a) > d(a)$, then delete ab from $E(G)$. If $\deg_{G-ab}(b) < d(b)$, then u.a.r. add an edge to b.



P-stability

The state space of the JS-chain: $\mathcal{G}'(d) := \mathcal{G}(d) \cup \bigcup_{i,j \in V} \mathcal{G}(d + \mathbb{1}_i + \mathbb{1}_j)$

To get a sample from $\mathcal{G}(d)$ in reasonable time by the JS-chain, we must have

Definition

$$|\mathcal{G}'(d)| \leq \mathrm{poly}(n) \cdot |\mathcal{G}(d)| \quad \forall d \in \mathcal{D}.$$

where $n = \dim(d)$. In this case, we call an infinite $\mathcal{D}$ a P-*stable* class of degree sequences.

Theorem (Jerrum and Sinclair 1990)

The JS-chain is rapidly mixing on degree sequences from a P-stable class.

Some P-stable regions (Jerrum, McKay, and Sinclair, 1989)

Theorem

The degree sequences $d \in [1, \Delta]^n$ *satisfying*

$$\Delta \leq 2\sqrt{n} - 2, \; d \in \mathbb{N}^n$$

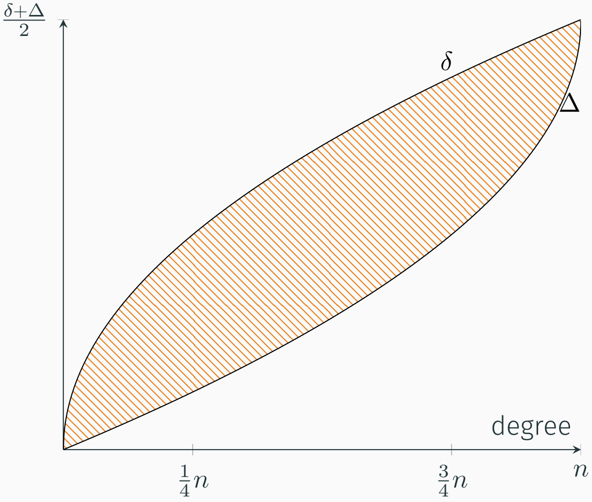
for any n *are* P*-stable.*

Theorem

The degree sequences d *satisfying*

$$(\Delta - \delta + 1)^2 \leq 4\delta(n - \Delta + 1), \; d \in \mathbb{N}^n$$

for any n *are* P*-stable. (See the plot on the right.)*



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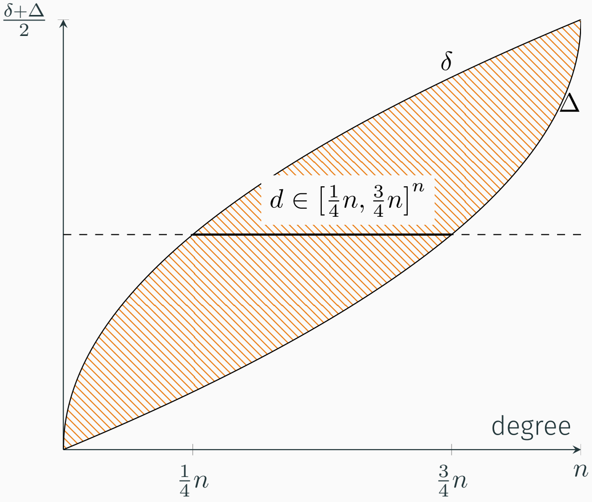

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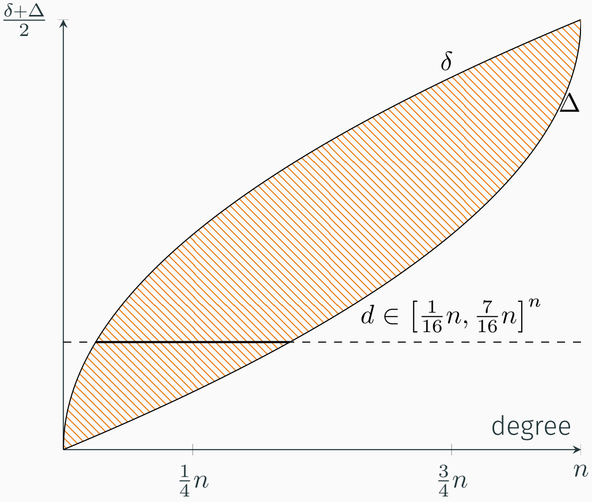

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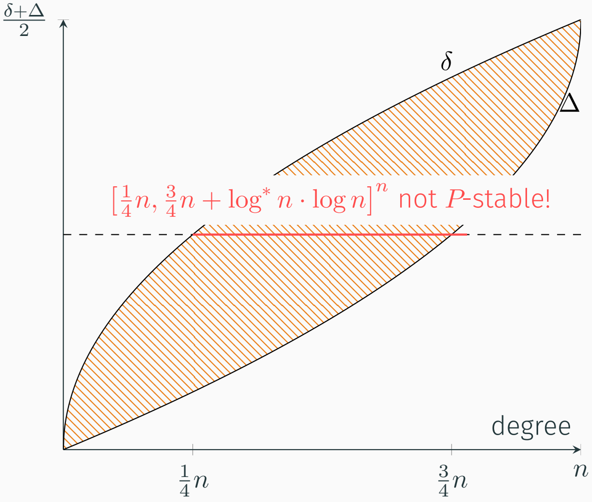


Switch Markov chains

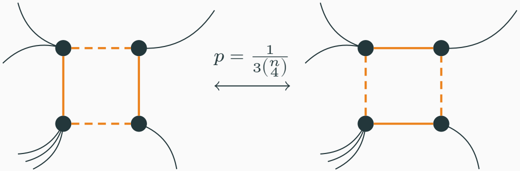
The switch Markov chain for simple and bipartite graphs

Proposed for bipartite graph by Kannan, Tetali, Vempala (1997)

State space: the set of realizations $\mathcal{G}(d)$ of a deg. seq. d

Transitions: exchange edges with non-edges along a randomly chosen alternating C_4 (least perturbation)

switch:



The switch Markov chain for directed graphs

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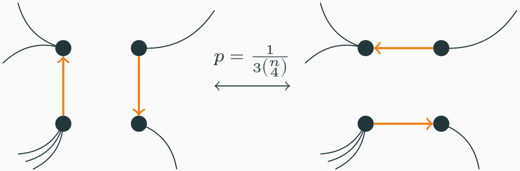

directed $\triangle$:

Previous and recent results

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$\Delta \leq c\sqrt{m}$	Greenhill 2015	Erdős, Miklós, M, Soltész 2018	

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Rapid mixing of the Switch Markov chain shown by

	simple	*bipartite*	*directed*
regular	Cooper, Dyer, Greenhill 2007	Erdős, Miklós, Soukup 2013	Greenhill 2011
$\Delta \leq c\sqrt{m}$	Greenhill 2015	Erdős, Miklós, M, Soltész 2018	
$[\delta, \Delta]$-type	Amanatidis and Kleer 2019	Erdős, Miklós, M, Soltész 2018	

Early results - bipartite degree sequences

PL Erdős, TRM, I Miklós, D Soltész (2018)

Theorem

Let d be a bipartite degree sequence. On the set of d satisfying

$$\Delta \leq \frac{1}{\sqrt{2}}\sqrt{m},$$

the switch Markov chain is rapidly mixing. (Moreover, the set is P-stable).

Theorem

Let d be a bipartite degree sequence on U and V as classes. On the set of d satisfying

$$(\Delta_V - \delta_V - 1)^+ \cdot (\Delta_U - \delta_U - 1)^+ \leq \max\left(\delta_V(|V| - \Delta_U), \delta_U(|U| - \Delta_V)\right),$$

the switch Markov chain is rapidly mixing. (Moreover, the set is P-stable).

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Theorem

Let d be a bipartite degree sequence on U and V as classes. On the set of d satisfying

$$(\Delta_V - \delta_V) \cdot (\Delta_U - \delta_U) \leq 4 \cdot \min\left(\delta_V(|V| - \Delta_U), \delta_U(|U| - \Delta_V)\right),$$

the switch Markov chain is rapidly mixing. (Moreover, the set is P-stable).

Early results - directed degree sequences

PL Erdős, TRM, I Miklós, D Soltész (2018)

Theorem

Let d be a directed degree sequence. On the set of d satisfying

$$\Delta_{\text{out}}, \Delta_{\text{in}} \leq \frac{1}{\sqrt{2}}\sqrt{m-4},$$

the switch Markov chain is rapidly mixing. (Moreover, the set is P-stable).

Theorem

Let d be a directed degree sequence. On the set of d satisfying

$$(\Delta_{\text{out}} - \delta_{\text{out}}) \cdot (\Delta_{\text{in}} - \delta_{\text{in}}) \leq \max\left(\delta_{\text{out}}(n - \Delta_{\text{in}} - 1), \delta_{\text{in}}(n - \Delta_{\text{out}} - 1)\right) + \mathcal{O}(n),$$

the switch Markov chain is rapidly mixing. (Moreover, the set is P-stable).

An unpublished result

PL Erdős, TRM, I Miklós, D Soltész (2018)

The proof of the theorems on the previous slides contain the main ideas to proving the following result:

Theorem (unpublished)

The switch Markov chain is rapidly mixing on bipartite and directed P-stable degree sequences.

Goal: extend the theorem to simple graphs

Unified method to prove rapid mixing of switch chains on P-stable degree sequences

Primitive circuit trails

PL Erdős, C Greenhill, TRM, I Miklós, D Soltész, L Soukup (2019+)

Definition

A closed walk $v_1v_2, v_2v_3, v_3v_4 \dots, v_{2k-1}v_{2k}, v_{2k}v_1$ (indices taken from $\mathbb{Z}/2k\mathbb{Z}$) which

- does not traverse the same edge twice and
- for any $i \neq j$ it satisfies $v_i = v_j \Leftrightarrow i \equiv j+1 \pmod 2$

is called a **primitive circuit trail**.

Definition

We say that C is an **alternating primitive circuit trail** in X, if

- $v_{2i-1}v_{2i} \notin E(X)$ and
- $v_{2i}v_{2i+1} \in E(X)$

for $1 \leq i \leq k$.

Unified method

PL Erdős, C Greenhill, TRM, I Miklós, D Soltész, L Soukup (2019+)

Let $\mathfrak{M}$ denote a graph model, e.g.: simple, bipartite, directed, etc.

Theorem

Let X be an arbitrary $\mathfrak{M}$ graph which contains an alternating primitive circuit trail C that traverses each of the vertices of X.

If for any such X there exists a sequence of switches transforming X into $X \triangle E(C)$ s.t.

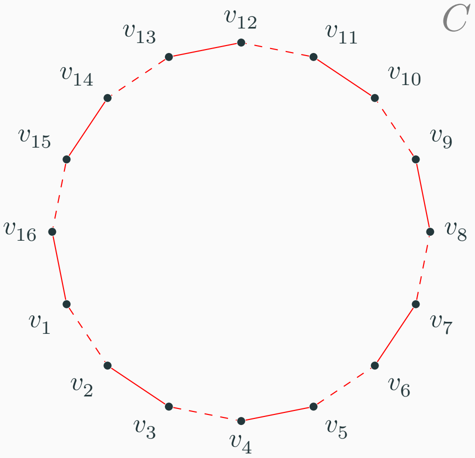
- *the length of the sequence is $\leq \mathrm{poly}_{\mathfrak{M}}(|V(X)|)$, and*
- *for any intermediate graph Z in the sequence and any $\mathfrak{M}$ graph Y s.t. $E(C) \subseteq E(X) \triangle E(Y)$, there exists a graph $Z' \in \mathcal{G}'(d(Y))$ such that $\ell_1(A_X + A_Y - A_Z, A_{Z'}) \leq c_{\mathfrak{M}}$*

then the switch Markov chain is rapidly mixing on P-stable $\mathfrak{M}$ degree sequences.

Bipartite graphs

PL Erdős, C Greenhill, TRM, I Miklós, D Soltész, L Soukup (2019+)

- $C = (v_1v_2, \dots, v_{15}v_{16}, v_{16}v_1)$
- Goal: find switch sequence $X, Z_1, Z_2, \dots, X \triangle E(C)$
- C is a bipartite primitive circuit $\implies C$ is a cycle



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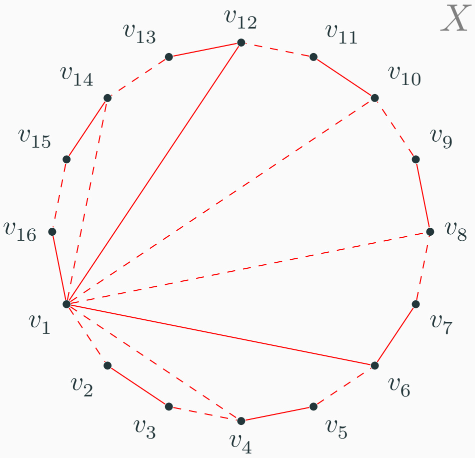

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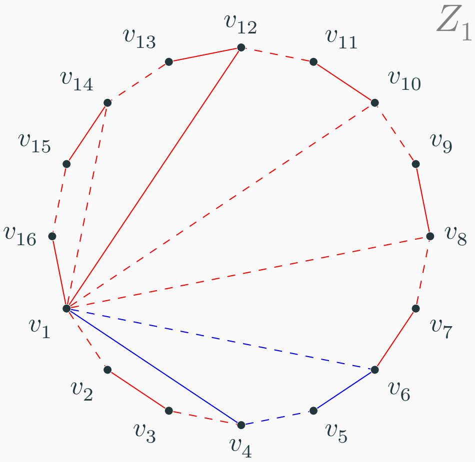

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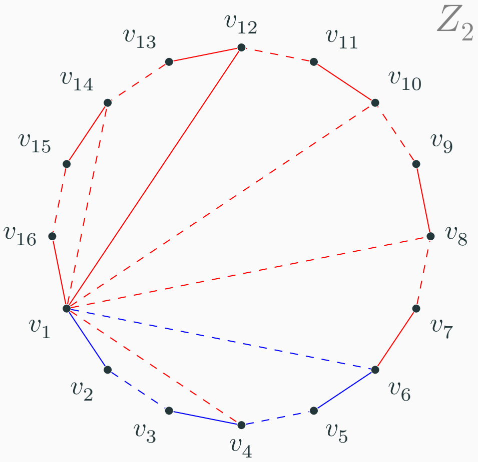

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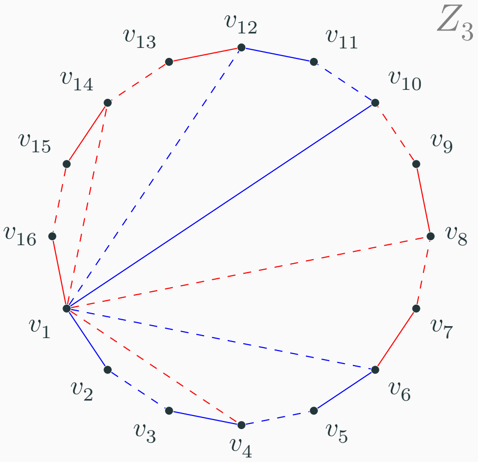

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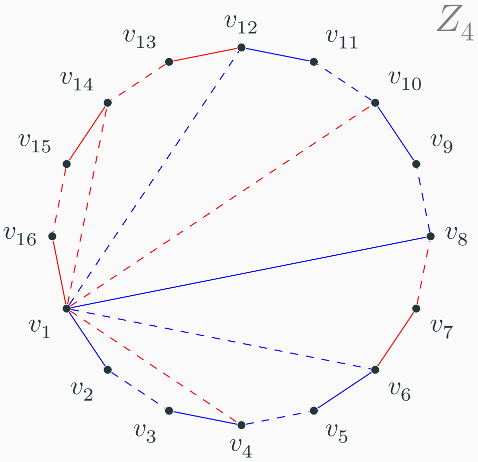

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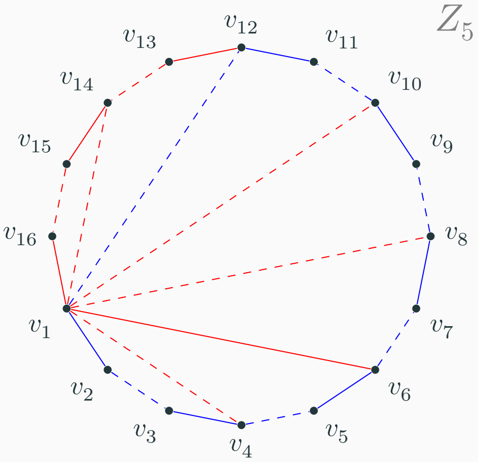

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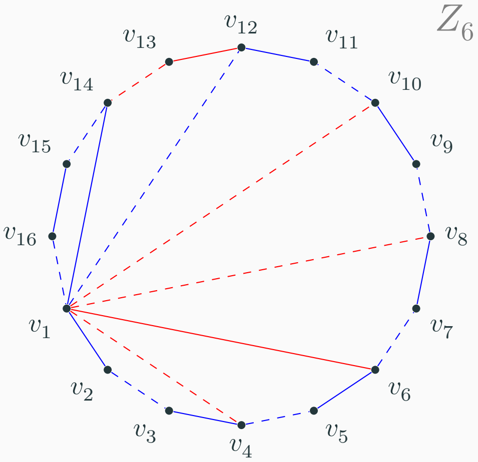

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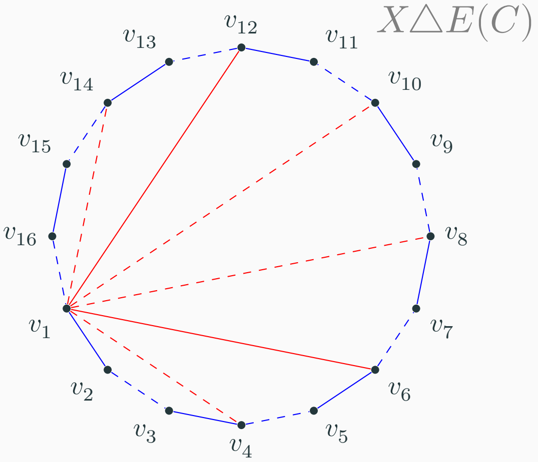

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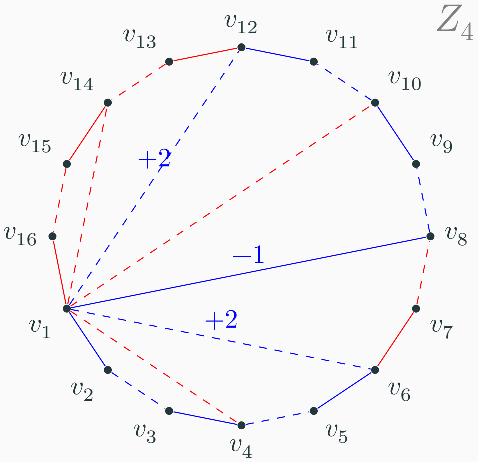
- Goal: find switch sequence $X, Z_1, Z_2, \dots, X \triangle E(C)$
- C is a bipartite primitive circuit $\implies$ C is a cycle
- red: original state in X, blue edge: flipped
- $A_X + A_Y - A_Z$: takes $0 - 1$ everywhere, except maybe at most two $+2$ on blue non-edge chords, and at most one -1 on a blue chord.



Coding - adjacency matrices

- $A_X + A_Y - A_Z$ is $0-1$ everywhere, except the first row
- The row of v_1 contains at most two $+2$ and at most one -1 entries

	$A_X + A_Y - A_Z$							
Vertices	v_2	v_4	...	$\cdots$	$\cdots$	v_{2j}	$\cdots$	v_{2k}
v_1	0	1	$\cdots$	$\cdots$	$\cdots$	2	$\cdots$	0
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\ddots$	$\vdots$	$\ddots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\ddots$	$\vdots$	$\ddots$	$\vdots$
v_{2k-1}				$\cdots$		$\cdots$		

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v_1	0	1	$\cdots$	$\cdots$	$\cdots$	2	$\cdots$	0
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\ddots$	$\vdots$	$\ddots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\ddots$	$\vdots$	$\ddots$	$\vdots$
v_{2k-1}				$\cdots$		$\cdots$		

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	$A_X + A_Y - A_Z$							
Vertices	v_2	v_4	$\dots$	$v_{2\ell}$	$\cdots$	v_{2j}	$\cdots$	v_{2k}
v_1	0	1	$\cdots$	$\cdots$	$\cdots$	2	$\cdots$	0
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\ddots$	$\vdots$	$\ddots$	$\vdots$
v_{2i-1}						0		$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\ddots$	$\vdots$	$\ddots$	$\vdots$
v_{2k-1}				$\cdots$		$\cdots$		
Min. row sum in v_1								

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Vertices	v_2	v_4	$\cdots$	$v_{2\ell}$	$\cdots$	v_{2j}	$\cdots$	v_{2k}
v_1	0	1	$\cdots$	0	$\cdots$	2	$\cdots$	0
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\wedge$	$\ddots$	$\vdots$	$\ddots$	$\vdots$
v_{2i-1}				1		0		$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\ddots$	$\vdots$	$\ddots$	$\vdots$
v_{2k-1}				$\cdots$		$\cdots$		

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Vertices	v_2	v_4	$\cdots$	$v_{2\ell}$	$\cdots$	v_{2j}	$\cdots$	v_{2k}
v_1	0	1	$\cdots$	-1	$\cdots$	2	$\cdots$	0
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\wedge$	$\ddots$	$\vdots$	$\ddots$	$\vdots$
v_{2i-1}				0		0		$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\ddots$	$\vdots$	$\ddots$	$\vdots$
v_{2k-1}				$\cdots$		$\cdots$		
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	$A_X + A_Y - A_Z$ +switch							
Vertices	v_2	v_4	$\cdots$	$v_{2\ell}$	$\cdots$	v_{2j}	$\cdots$	v_{2k}
v_1	0	1	$\cdots$	0	$\cdots$	1	$\cdots$	0
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\wedge$	$\ddots$	$\vdots$	$\ddots$	$\vdots$
v_{2i-1}				-1		1		$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\ddots$	$\vdots$	$\ddots$	$\vdots$
v_{2k-1}				$\cdots$		$\cdots$		
Min. row sum in v_1								

Coding - adjacency matrices

- $A_X + A_Y - A_Z$ is $0-1$ everywhere, except the first row
- The row of v_1 contains at most two $+2$ and at most one -1 entries
- With (at most two) switches and increasing the -1 entry by one, we turned $A_X + A_Y - A_Z$ into the adjacency matrix of some $Z' \in \mathcal{G}(d(Y) + \mathbb{1}(v_{2i-1}) + \mathbb{1}(v_{2\ell}))$

	$A_X + A_Y - A_Z$ +switch $+ v_{2i-1}v_{2\ell}$							
Vertices	v_2	v_4	$\cdots$	$v_{2\ell}$	$\cdots$	v_{2j}	$\cdots$	v_{2k}
v_1	0	1	$\cdots$	0	$\cdots$	1	$\cdots$	0
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\wedge$	$\ddots$	$\vdots$	$\ddots$	$\vdots$
v_{2i-1}				0		1		$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\ddots$	$\vdots$	$\ddots$	$\vdots$
v_{2k-1}				$\cdots$		$\cdots$		
Min. row sum in v_1								

Simple graphs

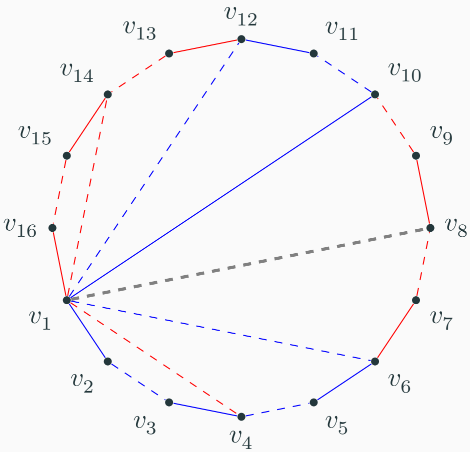
PL Erdős, C Greenhill, TRM, I Miklós, D Soltész, L Soukup (2019+)

C primitive circuit trail $\implies$

$$|\{v_{2i-1} \mid 1 \le i \le k\}| = k$$

$$|\{v_{2i} \mid 1 \le i \le k\}| = k$$

- Suppose $v_1 = v_8$
- red: original state in X, blue edge: flipped



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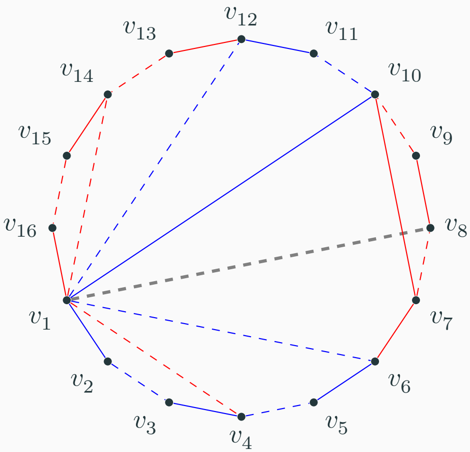

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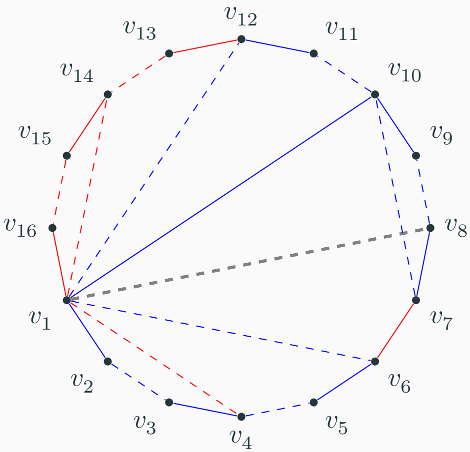

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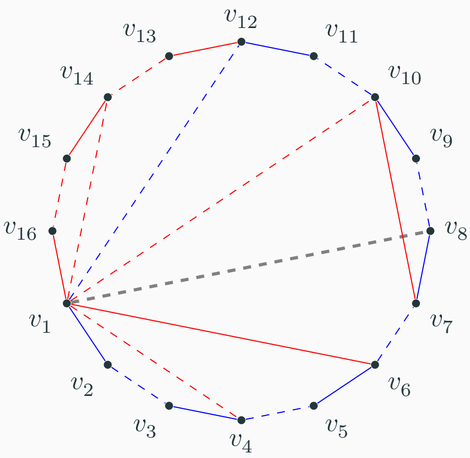

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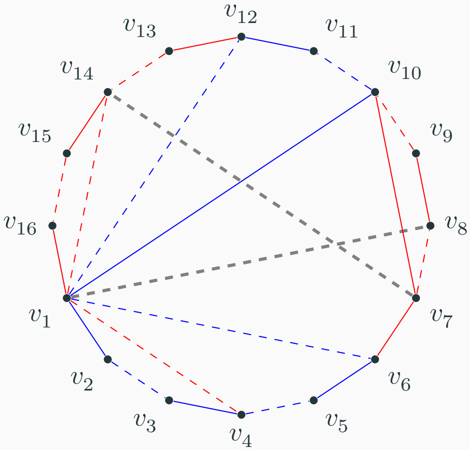

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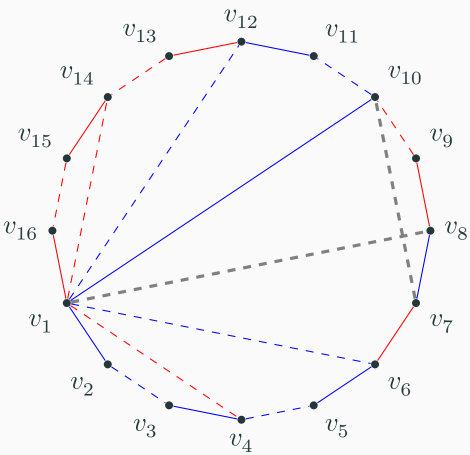

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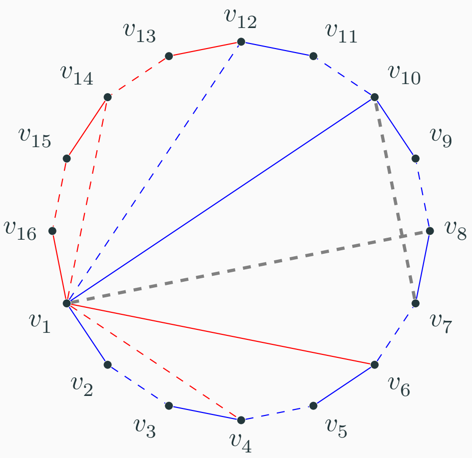
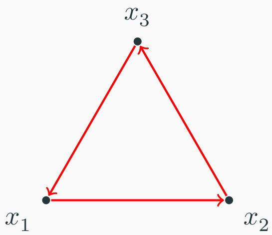


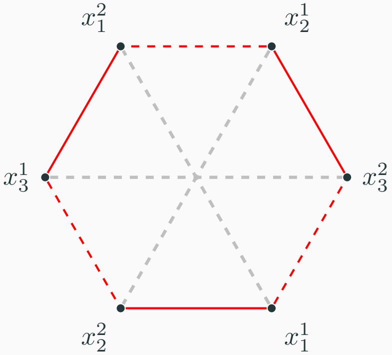
Directed graphs

Let $D = (\{v_1, \ldots, v_n\}, A)$ be directed graph. Let

$$X(D) = \left(\{v_i^1 \mid 1 \leq i \leq n\}, \{v_i^2 \mid 1 \leq i \leq n\}; \{v_i^1 v_j^2 \mid \overrightarrow{v_i v_j} \in A\}\right)$$

$$\mathcal{G}(\vec{d}) \longleftrightarrow \left\{G \in \mathcal{G}_{\text{bipartite}}\left(\vec{d}_{\text{in}}, \vec{d}_{\text{out}}\right) \,|v_i^1 v_i^2 \notin E(G)\right\}$$





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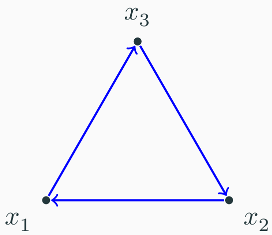

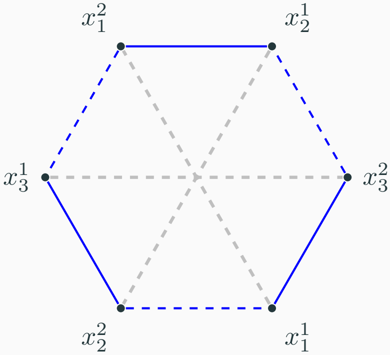

Comparison to other stability based results

Other notions of stability

Definition (strong stability)

$\exists k \; \forall d \in \mathcal{D} \; \forall d' \leq d$ s.t. $\|d' - d\|_1 \leq 2$ we have $\max_{G' \in \mathcal{G}(d')} \min_{G \in \mathcal{G}(d)} |E(G) \triangle E(G')| \leq k$

Theorem (Amanatidis and Kleer 2019)

The switch chain is rapidly mixing on ***strongly-stable*** *deg. sequences (simple, bipartite).*

The proof relies on the rapid mixing of the JS-chain (it seems the proof cannot be extended beyond P-stable).

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Definition (k-stability)

$$\forall d \in \mathcal{D}\ \forall d' \in \mathbb{N}^n \text{ s.t. } \|d' - d\|_1 \leq k \text{ we have } |\mathcal{G}(d')| \leq \mathrm{poly}(n) \cdot |\mathcal{G}(d)|$$

where $n = \dim(d)$. In this case, we call $\mathcal{D}$ a *k-stable* class of degree sequences.

Theorem (Gao and Greenhill 2020+)

The switch Markov-chain is rapidly mixing on ***8-stable*** *deg. sequences (simple, directed).*

Mixing times

Bounds on the mixing time of the switch chain for typical known rapidly mixing classes of simple degree sequences	
Amanatidis and Kleer (strongly stable)	$\tau(\varepsilon) \leq n^{48} \cdot (m \log 2m - \log \varepsilon)$
Gao and Greenhill (8-stable)	$\tau(\varepsilon) \leq n^{42} \cdot (m \log 2m - \log \varepsilon)$
P-stable	$\tau(\varepsilon) \leq n^{30} \cdot (m \log 2m - \log \varepsilon)$

- P-stability $\Leftrightarrow 2-$stability.
- Both strong stability and 8-stability imply P-stability.
- Almost all of the known rapid mixing regions are 8-stable and strongly-stable
- The above table tries to compare apples to oranges, the bounds are not verbatim quoted.

Heavy-tailed degree sequences

Suppose $d_1 \geq d_2 \geq \ldots \geq d_n$. Let $J(d) = \sum_{i=1}^{d_1} d_i$.

Theorem (Gao and Greenhill 2020+)

The set of degree sequences d satisfying

$$m(d) > J(d) + 9\Delta(d) + 23$$

is 8-stable (hence P-stable).

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The set of degree sequences d satisfying

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is both strongly-stable and P-stable.

These results contain deg. sequences that obey a power-law distribution-bound for $\gamma > 2$

Beyond P-stability...?

P-stability is not necessary for rapid mixing

For all $n, k \in \mathbb{Z}^+$, let us define the bipartite degree sequence

$$h_n := \begin{pmatrix} 1 & 2 & 3 & \cdots & n-2 & n-1 & n \\ n & n-1 & n-2 & \cdots & 3 & 2 & 1 \end{pmatrix}$$

Let the set of degree sequences which are k-close to h in ℓ_1-norm be

$$B_k(d) = \left\{ d' \in \mathbb{N}^n \;\middle|\; \|d' - d\|_1 \le k \right\}$$

Theorem (PL Erdős, E Győri, TRM, I Miklós, D Soltész 2020+)

For any $c \in \mathbb{R}^+$, the switch Markov chain is rapidly mixing on the non-P-stable class

$$\bigcup_{k=1}^{\infty} B_{c\sqrt{\log n}}(h_n)$$

Thank you for attending my ZOOM presentation!

Homepage: `https://trm.hu`

Full papers

`https://doi.org/10.1371/journal.pone.0201995`

`https://arxiv.org/abs/1903.06600`

`https://arxiv.org/abs/1909.02308`

Second moment

Theorem (Svante Janson 2006)

Let $(G_n)_{n=1}^{\infty}$ be a sequence of random multigraphs generated by the configuration model, such that $e(G_n) = \Theta(n)$. Then

$$\liminf_{n\to\infty} \Pr(G_n \text{ is simple}) \Leftrightarrow \sum_{v\in V(G_n)} d_{G_n}(v)^2 = \mathcal{O}(n)$$

Theorem (Svante Janson 2020)

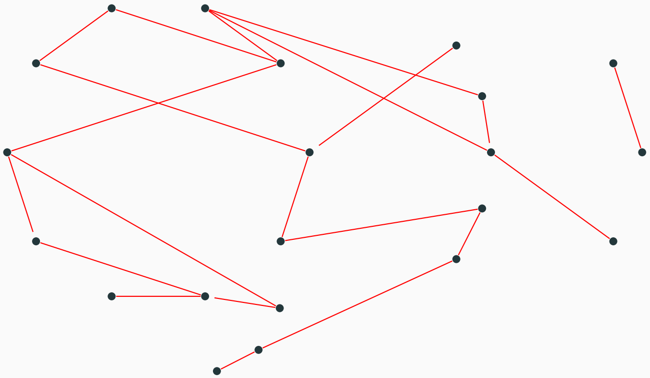
By randomly switching, TV dist goes to 0.

Proof outline of rapid mixing on P-stable degree sequences

- Use the Jerrum-Sinclair result: construct a multicommodity-flow that sends a 1-flow between any two realizations in the Markov-graph such that no realization is overloaded

- Determining a flow between any two $X, Y \in \mathcal{G}(d)$
  - Decompose $E(X)\Delta E(Y)$ into red/blue alternating circuit trails: the red and blue degrees are the same in $E(X)\Delta E(Y)$, because X and Y share the same degree sequence.
  - Decompose alternating circuit trails into primitive alternating circuit trails
  - Process primitive circuits: exchange edges with non-edges via the previous algorithm

Decomposing the symmetric difference $E(X)\Delta E(Y)$



- Let s be a complete matching between the red and blue edges at each vertex
- Thus $E(X)\triangle E(Y) = W_1 \uplus \ldots \uplus W_k$, where each W_i is an alternating-circuit
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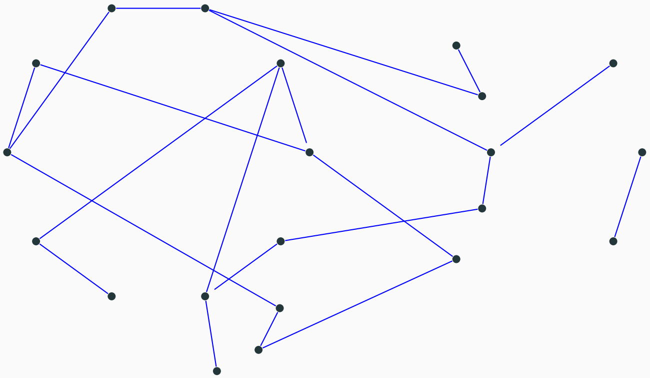

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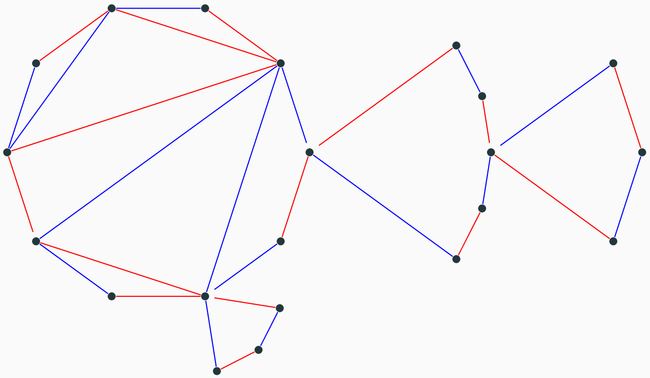

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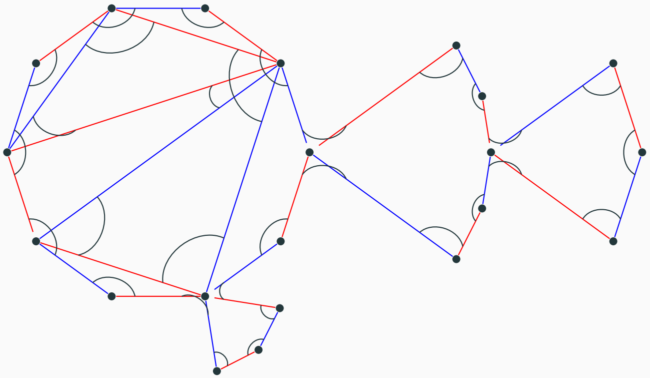

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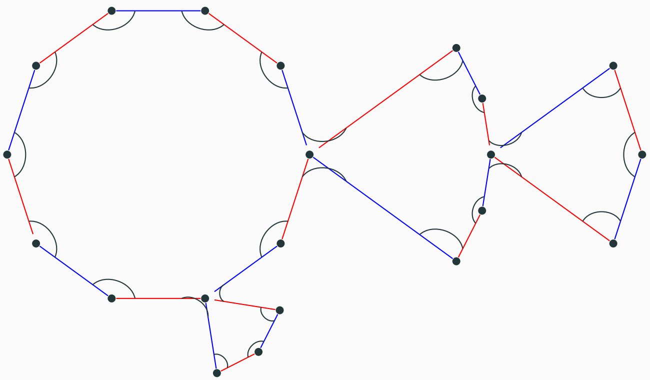

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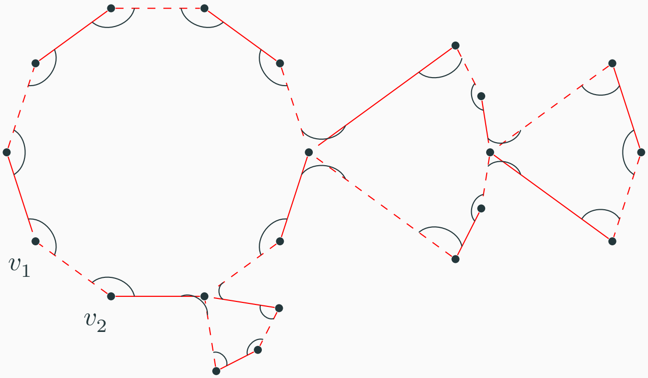

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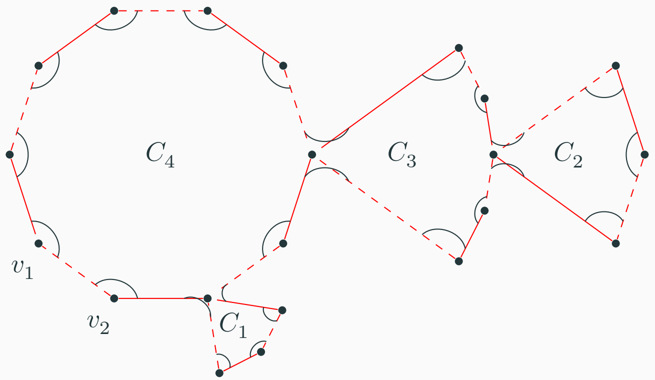

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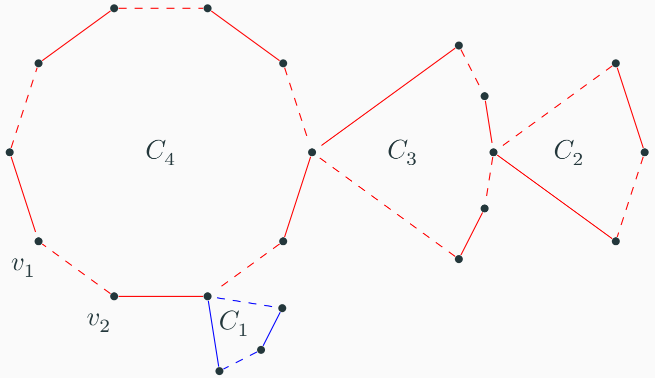

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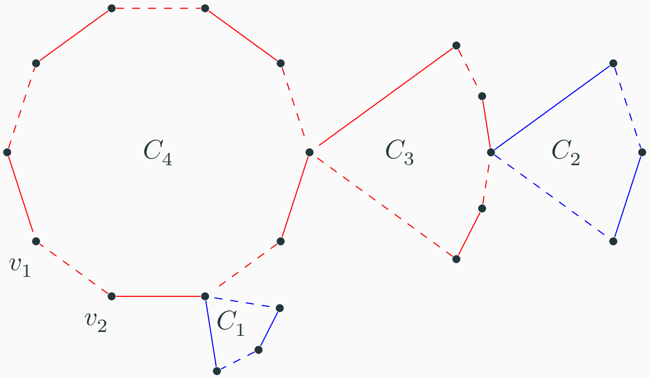

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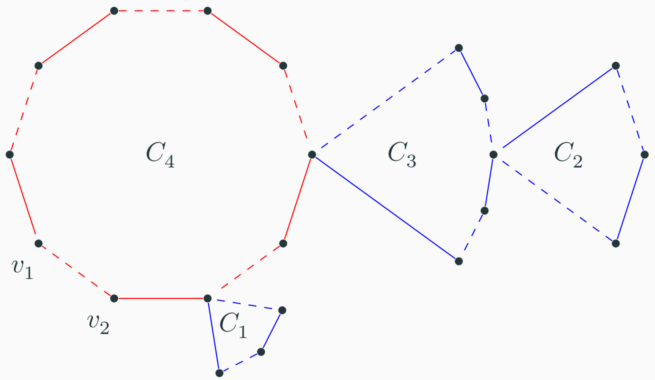

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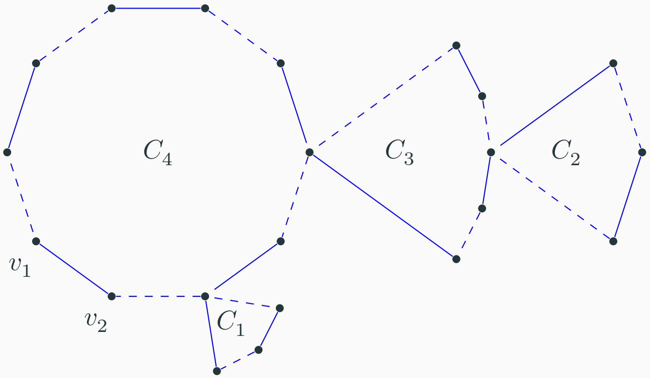

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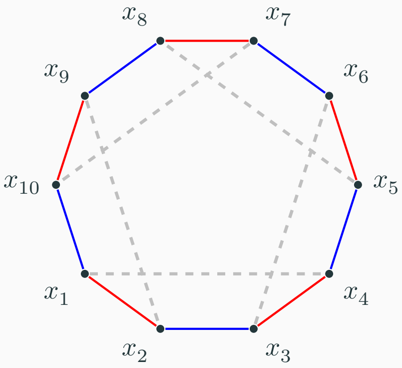
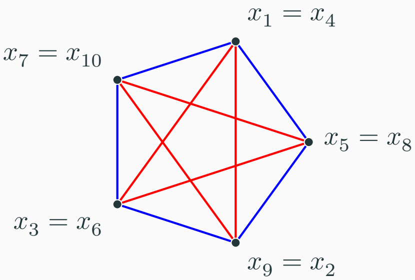
Sweeping primitive alternating circuits - demo on an extra special case





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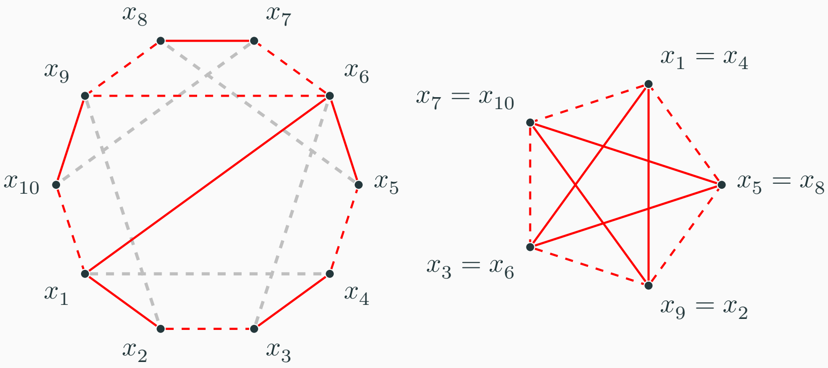

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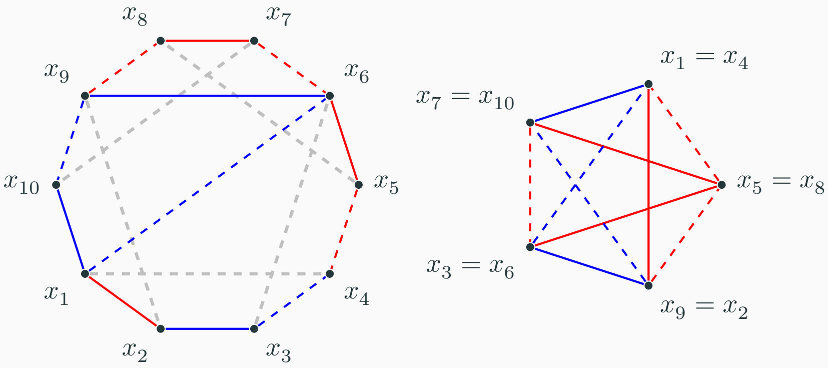

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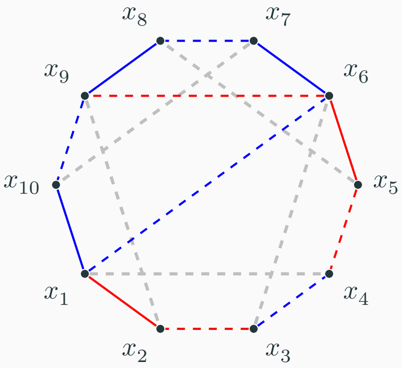

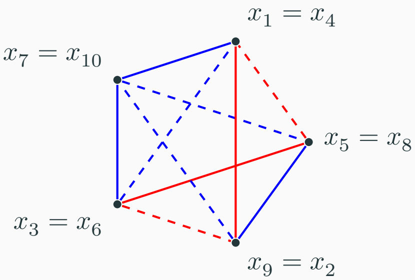

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